15th Australasian Fluid Mechanics Conference
The University of Sydney, Sydney, Australia
13-17 December 2004

The effect of zero-mean suction on Görtler vortices: a receptivity approach

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Abstract

We consider the effect of zero-mean suction on the development of Görtler vortices in the boundary layer flow over a concavely curved surface. The zero-mean suction is assumed not to affect the basic boundary-layer flow and so can be modelling by modifying the impermeability condition at the surface to read $v = \delta \cos \omega t$ where $0 < \delta \ll 1$. The problem is posed in terms of vortex receptivity and we demonstrate that small amplitude zero-mean suction with a frequency ω satisfying $\omega > 2.254G^{2/5}$ (where G is the Görtler number) serves to fully stabilise the most unstable Görtler vortex.

Introduction

The use of boundary layer suction to control the process of transition to turbulence is one which has a long history in the scientific literature; this is reviewed in Schlichting [14]. Although the majority of earlier work on suction control has focused upon flat-plate boundary layers, and consequently on controlling the growth of Tollmien-Schlichting waves, there have been a number of studies on the effect of suction on centrifugal instabilities (that is, Görtler vortices). Görtler vortices arise in the boundary-layer flow over a concavely curved surface and are particularly relevant in the design of laminar flow aerofoils. For example, the laminar flow wing considered by Mangalam *et al.* [11] had appreciable regions of concave curvature on the underside of the airfoil. The introduction of concave curvature then presented the potential problem of an earlier transition to turbulence due to the development of a secondary instability on the streamwise aligned counter-rotating Görtler vortices.

The majority of theoretical work on the effect of suction on Görtler vortices has focused upon the asymptotic suction profile

$$\overline{u} = 1 - e^{-v_s y}, \quad \overline{v} = -v_s. \tag{1}$$

Kobayashi [9] and Floryan & Saric [7] showed that suction serves to stabilise Görtler vortices; the latter work demonstrating that a larger level of suction is required to stabilise Görtler vortices than is required to stabilise Tollmien-Schlichting waves. A similar result was reported by Lin & Hwang [10] in their computational study Görtler vortices on a heated concave surface. Myose & Blackwelder [12] undertook a series of experiments using isolated suction holes (placed in the low-speed region between the counter-rotating Görtler vortices) and were able to demonstrate that this method required two orders of magnitude less suction to control the breakdown of Görtler vortices, over a comparable area, when compared to an asymptotic suction profile approach. However, this study was concerned with controlling the secondary instability that occurs in Görtler vortex flows through the modification of the low-speed flow region and so did not suppress the development of the Görtler vortices themselves but controlled the onset of secondary instability.

Park & Huerre [13] also employed an asymptotic suction profile (given by (1) with $v_s = 0.5$) in their study of the nonlinear development and subsequent secondary instability of Görtler vortices. Their work was not concerned with suction control but was aimed at exploiting the fact that with a base flow given by (1) all issues regarding non-parallelism in the development of Görtler vortices could be side-stepped. It was the early work of Hall [8] that was the first to emphasise the fact that the Görtler vortex instability is crucially linked to the non-parallel evolution of the boundary-layer flow. Further recent work on suction control of Görtler vortices can be found in Balakumar & Hall [3].

Recently Denier [4] has reconsidered the stability of the asymptotic suction profile (1) in order to determine the level of suction required to fully stabilise the flow to Görtler vortices. By focusing on the most unstable Görtler vortex mode (whose structure is described in [6], [15]) it can be shown that suction will fully stabilise the flow when the level of suction, measured by v_s in (1), satisfies $v_s > 0.3581G^{1/3}$. Here G is the Görtler number, defined in (3).

In the past few years there has been considerable attention given to the problem of unsteady suction at the leading edge of an aerofoil. Here we consider the effect of an unsteady suction (that is, alternating suction and blowing) on the stability of Görtler vortices. We will assume that the suction profile has a **zero-mean** state; in other words we will assume that the suction velocity is prescribed at the surface and is proportional to $\cos \omega t$.

Formulation

The equations governing the *linear* evolution of span-wise periodic disturbances to a boundary layer flowing over a curved surface are

$$\tilde{U}_x + \tilde{V}_y + ik\tilde{W} = 0, \tag{2a}$$
$$\tilde{U}_{yy} - k^2\tilde{U} = \overline{u}\tilde{U}_x + \tilde{U}\overline{u}_x + \overline{v}\tilde{U}_y + \tilde{V}\overline{u}_y, \tag{2b}$$
$$\tilde{V}_{yy} - k^2\tilde{V} = \tilde{P}_y + G\chi\tilde{U}\overline{u} + \overline{u}\tilde{V}_x + \tilde{U}\overline{v}_x + \overline{v}\tilde{V}_y + \tilde{V}\overline{v}_y, \tag{2c}$$
$$\tilde{W}_{yy} - k^2\tilde{W} = ik\tilde{P} + \overline{u}\tilde{W}_x + \overline{v}\tilde{W}_y, \tag{2d}$$

where $2\pi/k$ is the spanwise-wavelength of the disturbance, an over-bar denotes a basic boundary-layer variable, a tilde denotes a disturbance quantity, x is the streamwise coordinate and y the usual boundary-layer variable. The precise form for the basic boundary-layer is relatively unimportant in what follows. We will simply assume that the boundary layer remains attached - Görtler vortices in separated flows were described by Denier & Bassom [5]. In what follows we take $(\overline{u}, \overline{v})$ to be the streamwise and vertical velocity components within an attached boundary layer and so governed by Prandtl's boundary-layer equations

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0,$$
$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial u}{\partial x} = -\frac{\partial \overline{p}}{\partial x} + \frac{\partial^2 \overline{u}}{\partial y^2},$$

where $\overline{p}_x = u_e(x)u_{ex}(x)$ denotes the streamwise pressure gradient. These must be solved subject to no-slip boundary conditions $\overline{u} = \overline{v} = 0$ on $y = 0$ and $\overline{u} \to u_e$ as $y \to \infty$.

The important parameter appearing in (2) is the Görtler number G which is traditionally defined according to

$$G\chi = Re^{1/2} g_{xx} \qquad (3)$$

where $y = g(x)$ denotes the position of the wall and thus g_{xx} is the wall curvature, which is positive if the surface is concavely curved, and Re is the Reynolds number. For boundary-layer flows over a surface with even a moderate level of curvature the Görtler number is typically large due to the presence of the Reynolds number factor appearing in (3). Thus most boundary-layer applications involving Görtler vortices can typically be described as large Görtler number flows. This fact allows some considerable simplification of the fully parabolic system of equations (2) as has been described by Hall [8]. More importantly, it is the large Görtler number limit the most unstable vortex mode occurs (see Denier *et al.* [6] and Timoshin [15] for details). The wavelength of this mode scales as $O(G^{-1/5})$ and it is confined to within a viscous layer of thickness $O(G^{-1/5})$ situated at the wall. Furthermore the streamwise growth rate has magnitude $O(G^{3/5})$ and n this regime the maximum growth rate of all vortex-like perturbations occurs.

Turning to the question of the physical boundary conditions appropriate to the flow we focus our attention on the problem of zero-mean suction (alternatively blowing) at the surface. By zero-mean we take to mean blowing whose time averaged behaviour shows no mean component and is therefore assumed to have no effect upon the mean-boundary layer flow. We will model this by prescribing the wall-normal perturbation velocity to be given by

$$\tilde{V} = \begin{cases} 0 & \text{if} \quad x < \overline{x} \\ \delta F\left(J(x-\overline{x})\right)\exp\left(ikz + i\omega t\right) & \text{if} \quad x \geq \overline{x} \end{cases} \qquad (4)$$

where J is a constant that determines the streamwise extent of the active region of suction/blowing, $2\pi/\omega$ is the suction frequency and $0 < \delta \ll 1$ is a small parameter which sets the strength of the suction. We are therefore focusing on small amplitude perturbations to the basic boundary-layer flow induced by small amplitude blowing/suction. We have also assumed that the region of active blowing is spanwise periodic[1]; this is equivalent to assuming that there is a periodic array of finite suction slots located at $x = \overline{x}$.

In addition to this condition on $\tilde{V}$ we must also impose the usual no-slip boundary conditions on the streamwise and spanwise velocity perturbations

$$\tilde{U} = \tilde{W} = 0 \quad \text{on} \quad y = 0,$$

and the condition that the perturbation is confined to within the boundary layer

$$(\tilde{U}, \tilde{V}, \tilde{W}) \to 0 \quad \text{as} \quad y \to \infty.$$

Vortex receptivity

We focus our attention on the receptivity of the most unstable Görtler vortex to zero-mean suction. As noted earlier the most unstable Görtler vortex has a streamwise growth rate of $O(G^{3/5})$ and is confined to an $O(G^{-1/5})$ thick layer located at the wall. We therefore introduce a new stretched wall coordinate $\varphi = G^{1/5}y$ (where y is the usual boundary-layer coordinate). Led by the results of Denier *et al.* [6] and Timoshin [15] we consider perturbations to the basic flow in the form

$$(u,v,w,p) = (\overline{u}, Re^{-1/2}\overline{v}, 0, \overline{p}) + \delta(G^{-2/5}u_1, v_1, w_1, G^{1/5}p_1) \times \exp\left(ikz + G^{3/5}\int \beta(x)\mathrm{d}x + i\omega t\right),$$

where δ is the (infinitesimally) small perturbation amplitude of the zero-mean suction/blowing. In these expansions we have anticipated that the disturbance to the basic flow is of the same order of magnitude as the zero-mean suction/blowing velocity and so is of size $O(\delta)$. This fixes the amplitude of the vertical velocity perturbation term; the relative magnitude of the other terms is then a simple consequence of balancing terms in the continuity and momentum equations.

In order to determine the frequency at which the suction/blowing first affects the stability of the flow we must necessarily balance

$$\frac{\partial u}{\partial t} \sim \overline{u}\frac{\partial u}{\partial x}. \qquad (5)$$

Within the viscous sub-layer the mean streamwise velocity expands as

$$\overline{u} = \mu y + \cdots = \mu G^{-1/5}\varphi + \ldots,$$

where $\mu = \overline{u}'(x,0)$ is the wall shear; given our previous comments on the boundary layer remaining attached, μ is taken to be positive. Taken with the fact that the streamwise growth rate of the most unstable Görtler vortex is $O(G^{3/5})$ the balance expressed by (5) implies that $\omega = O(G^{2/5})$ (or equivalently the frequency of the blowing/suction must be $O(G^{-2/5})$). Thus it will be the, relatively, low frequency suction (through the periodic pumping) that will affect the stability of the flow.

In order to pose this problem in the form of a flow *receptivity* problem we suppose that the suction velocity is given by

$$v = \begin{cases} 0 & \text{if} \quad x < \overline{x} \\ F\left(\tilde{J}G^{3/5}(x-\overline{x})\right)\exp\left(i\lambda G^{1/5}z + i\omega t\right) & \text{if} \quad x \geq \overline{x} \end{cases} \qquad (6)$$

where we have set the vortex wavenumber $k = \lambda G^{1/5}$ thus allowing us to focus upon the wavenumber regime containing the most unstable Görtler vortex. To simply matters let us define $\tilde{x} = \tilde{J}G^{3/5}(x-\overline{x})$ and write

$$\begin{aligned} u &= u_0(\tilde{x},\varphi) + G^{-1/5}u_1(\tilde{x},\varphi) + \cdots, \\ (\chi\mu^2 G)^{-3/5} v &= v_0(\tilde{x},\varphi) + G^{-1/5}v_1(\tilde{x},\varphi) + \cdots, \end{aligned}$$

Setting $\omega = G^{2/5}\omega_0$ and substituting our expansions into the governing equation yields, to $O(\delta)$, the system of equations (in canonical form)

$$\left(\frac{\partial^2}{\partial\varphi^2} - \frac{\varphi}{\tilde{\lambda}^3}\frac{\partial}{\partial\tilde{x}} - 1 - \frac{i\tilde{\omega}}{\tilde{\lambda}^2}\right)\left(\frac{\partial^2}{\partial\varphi^2} - 1\right)\tilde{V}_0 = -\frac{\varphi\tilde{U}_0}{\tilde{\lambda}^2}, \qquad (7a)$$

$$\left(\frac{\partial^2}{\partial\varphi^2} - \frac{\varphi}{\tilde{\lambda}^3}\frac{\partial}{\partial\tilde{x}} - 1 - \frac{i\tilde{\omega}}{\tilde{\lambda}^2}\right)\tilde{U}_0 = \frac{\tilde{V}_0}{\tilde{\lambda}^2}, \qquad (7b)$$

which must be solved subject to the boundary conditions

$$\begin{aligned} &\tilde{U}_0 = \tilde{V}_0' = 0, \quad \tilde{V}_0 = F(\tilde{x}) \qquad \text{on} \qquad \varphi = 0, \\ &\tilde{U}_0, \tilde{V}_0, \tilde{V}_0' \to 0 \quad \text{as} \quad \varphi \to \infty. \end{aligned}$$

Here $\tilde{\lambda}$ and $\tilde{\omega}$ are the scaled wavenumber and frequency, respectively.

[1] This assumption is not necessary. Indeed, a suction slot with a finite extent in *both* the streamwise and spanwise direction can be dealt with by simply taking the Fourier transform in z. This however unduly complicates the subsequent analysis and so we choose not to consider this problem here.

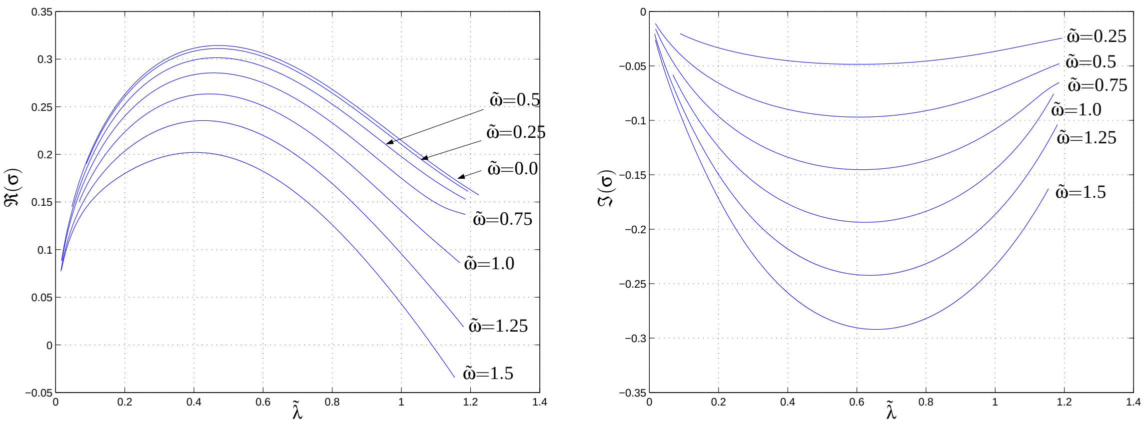

Figure 1: Plot of the first eigenvalue of system (8) for a variety of values of $\tilde{\omega}$. Shown is the (a) real part and (b) imaginary part of σ versus $\tilde{\lambda}$.

System (7) is most readily solved by taking the Laplace transform with respect to x. To this end we define

$$\hat{U}_0 = \int_0^\infty e^{-\sigma\tilde{x}}\tilde{U}_0(\varphi,\tilde{x})\mathrm{d}\tilde{x}, \quad \hat{V}_0 = \int_0^\infty e^{-\sigma\tilde{x}}\tilde{V}_0(\varphi,\tilde{x})\mathrm{d}\tilde{x}.$$

and upon taking the Laplace transform of system (7) we obtain

$$\left(\frac{\partial^2}{\partial\varphi^2} - \frac{\sigma\varphi}{\tilde{\lambda}^3} - 1 - \frac{i\tilde{\omega}}{\tilde{\lambda}^2}\right)\hat{U}_0 = \frac{\hat{V}_0}{\tilde{\lambda}^2}, \tag{8a}$$

$$\left(\frac{\partial^2}{\partial\varphi^2} - \frac{\sigma\varphi}{\tilde{\lambda}^3} - 1 - \frac{i\tilde{\omega}}{\tilde{\lambda}^2}\right)\left(\frac{\partial^2}{\partial\varphi^2} - 1\right)\hat{V}_0 = -\frac{\varphi\hat{U}_0}{\tilde{\lambda}^3}, \tag{8b}$$

which must be solved subject to the boundary conditions

$$\hat{U}_0 = \hat{V}_0' = 0, \quad \hat{V}_0 = \hat{F}(\sigma) \quad \text{on} \quad \varphi = 0, \tag{8c}$$

$$\hat{U}_0, \hat{V}_0, \hat{V}_0' \to 0 \quad \text{as} \quad \varphi \to \infty, \tag{8d}$$

where $\hat{F}(\sigma)$ is the transform of the function $F(\tilde{x})$. As noted by Denier *et al.* [6] this system has solutions which possess simple poles at $\sigma = \sigma_j$ where σ_j is the jth eigenvalue of the homogeneous system (8), $j = 1, 2, \ldots$. We therefore seek a solution to (8) in the form

$$(\hat{U}_0, \hat{V}_0) = \frac{\Delta_j(U_{0j}(\sigma_j,\varphi), V_{0j}(\sigma_j,\varphi))}{(\sigma - \sigma_j)} + (U_{1j}(\sigma_j,\varphi), V_{1j}(\sigma_j,\varphi)) + \cdots. \tag{9}$$

Substitution into (8) shows that U_{0j}, V_{0j} are the eigenfunctions of (8) corresponding to eigenvalue $\sigma = \sigma_j$. We will normalise these functions so that U_{0j} has a maximum value of unity. At next order we find that the functions U_{1j}, V_{1j} satisfy an inhomogeneous form of (8). Such an inhomogeneous equation only has a solution provided a solvability condition on the inhomogeneous terms is satisfied. In our case this solvability condition serves to determine the *receptivity coefficient* Δ_j as

$$\Delta_j = \frac{\tilde{\lambda}^3 Q_2'''(0)}{\int_0^\infty \varphi\left[Q_1 U_{0j} + Q_2\left(V_{0j}'' - V_{0j}\right)\right]\mathrm{d}\varphi} \tag{10}$$

where Q_1 and Q_2 are the adjoint eigenfunctions satisfying the system

$$\left(\frac{\partial^2}{\partial\varphi^2} - \frac{\sigma\varphi}{\tilde{\lambda}^3} - 1 - \frac{i\omega}{\tilde{\lambda}^2}\right)Q_1 + \frac{\varphi}{\tilde{\lambda}^3}Q_2 = 0, \tag{11a}$$

$$\left(\frac{\partial^2}{\partial\varphi^2} - 1\right)\left(\frac{\partial^2}{\partial\varphi^2} - \frac{\sigma\varphi}{\tilde{\lambda}^3} - 1 - \frac{i\omega}{\tilde{\lambda}^2}\right)Q_2 - \frac{1}{\tilde{\lambda}^2}Q_1 = 0, \tag{11b}$$

subject to the boundary conditions

$$Q_1 = Q_2 = Q_2' = 0 \quad \text{on} \quad \varphi = 0, \tag{11c}$$

$$Q_1, Q_2, Q_2' \to 0 \quad \text{as} \quad \varphi \to \infty. \tag{11d}$$

In order to complete the problem we must invert the transformed velocity field; to do this we must be more precise about the form of the function F appearing in (6). If we are interested in the effect of an isolated suction slot we can take F to be a function of compact support in which case F will not have any singularities in $\sigma_r \geq 0$. Then, as discussed above, the only singularities of (8) in $\sigma_r \geq 0$ correspond to the simple poles discussed above the contour of integration for the inverse transform can be chosen parallel to the imaginary axis to the right of $\sigma = \Re(\sigma_1)$. The contour is then closed in the left-hand half plane $\Re(\sigma) < \Re(\sigma_1)$ and the only contribution to the inverse Laplace transform then comes from the simple poles at $\sigma = \sigma_j$. Thus we obtain

$$(u_0, v_0) = \sum_{j=1}^{\infty}(U_{0j}, V_{0j})\Delta_j\hat{F}(\sigma_j)e^{\sigma_j x} \tag{12}$$

so that for a given value of $\tilde{\lambda}$ the effective coupling coefficient between the isolated (in x) unsteady suction and the vortex field is $\Delta_j\hat{F}(\sigma_j)$.

Results

The eigenvalue problem posed by (8) was solved by first discretising the system using second order accurate centred differences in φ. The homogeneous boundary condition $\hat{U}_0$ on $\varphi = 0$ was replaced with the normalisation condition $\hat{U}_0' = 1$ on $\varphi = 0$. When these boundary conditions are implemented in the discretised version of (8) an inhomogeneous matrix equation is obtained which can readily be solved. Newton iteration is then performed on the eigenvalue ω until the final boundary condition $\hat{U}_0 = 0$ on $\varphi = 0$ is satisfied, to within some predefined tolerance. An identical technique is used to solve the adjoint system (11); by noting that the adjoint system has the same eigenvalues as system (8) we have a useful check on the accuracy of our results.

The eigenvalues of system (8) are presented in figure 1 for a variety of values of the forcing frequency $\tilde{\omega}$. These show similar trends to those discussed in Bassom & Hall's [1] work on the effect of crossflow on Görtler vortices. In particular we note that for increasing frequency the vortices become stabilised at some finite value of the scaled wavenumber $\tilde{\lambda}$. Additionally

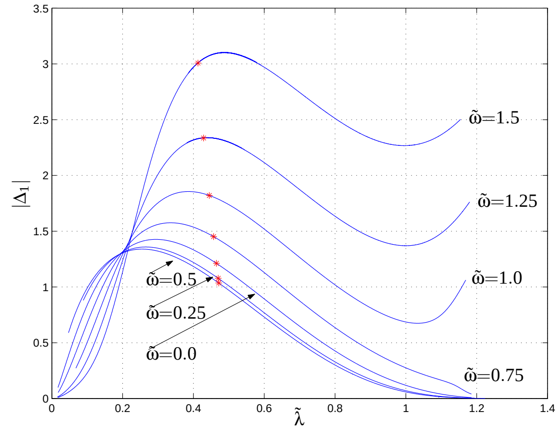

Figure 2: Plot of the magnitude of the receptivity coefficient $|\Delta_1|$ for the first eigenvalue of system (8) versus wavenumber $\tilde{\lambda}$. The asterisk indicates the wavenumber location of the maximum growth rate (see Fig. 1a).

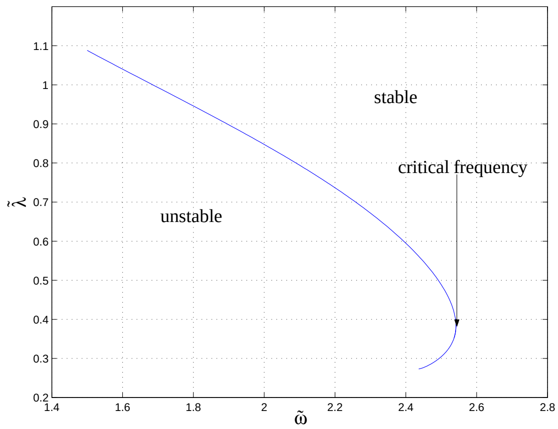

Figure 3: Plot of neutral wavenumber $\tilde{\lambda}$ (at which $\Re(\sigma) = 0$) as a function of the imposed frequency $\tilde{\omega}$.

the magnitude of the largest growth rate $\Re(\sigma)$ decreases with increasing frequency $\tilde{\omega}$.

The receptivity coefficient for the leading order eigenmode, $|\Delta_1|$, is given in figure 2. In order to interpret these results the values of the receptivity coefficient must be considered in the context of the streamwise response of the vortex disturbance which is given by (12). Thus the streamwise growth of the vortex is determined by the real part of σ. As figure 1 demonstrates increasing the frequency serves to reduce the growth rate. For sufficiently high frequencies the flow is completely stabilised, as is demonstrated by the *neutral curve* presented in figure 3. To the left of this curve the flow is unstable (over a finite band of vortex wavenumbers). The turning point in this curve is highlighted and occurs at $\tilde{\omega} \approx 2.2542$. Remembering that our analysis has focused upon the most unstable Görtler vortex we can then conclude that zero-mean suction of a frequency greater than $2.2542G^{2/5}$ will stabilise the flow to Görtler vortices (or more precisely, promote spanwise periodic disturbances which decay downstream).

Conclusions

We have shown that zero-mean suction can promote growing Görtler vortices in the boundary-layer flow over a concavely curved surface. If the frequency of the zero-mean suction is suitably high the vortices will decay immediately downstream of the source of the suction. Thus zero-mean suction provides a potential mechanism for the suppression of Görtler vortices.

Acknowledgements

Thanks to S.R. Otto (R&A Rules Limited) and J.A.K. Stott (Bradford College) for some useful discussions on this problem. This work was funded by the Australian Research Council through Discovery Grant DP0210877.

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