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Shedding Some Light on β-factors

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Abstract

The use of a Fickian (infinitesimal–mixing–length) framework for the case of turbulent mixing can necessitate the use of ad hoc modifications (e.g. β–factors) in order to reconcile experimental data with theoretical expectations. This is because in many cases turbulent mixing occurs on scales which cannot be considered infinitesimal. In response to this problem a Finite–Mixing–Length (FML) model for turbulent mixing was derived by Nielsen and Teakle [10]. This paper considers the application of this model to the scenario of suspended sediment in steady, uniform channel flows. It is shown that, unlike the Fickian framework, the FML model is capable of explaining why β–factors are required to be an increasing function of w_s/u_*. The FML model does not on its own explain observations of $\beta < 1$, seen in some flat–bed experiments. However, some potential reasons for $\beta < 1$ are considered.

Introduction

Fickian (gradient) diffusion is the usual theoretical framework applied to modelling the turbulent suspension of sediment. However, measurements of actual suspended sediment profiles in the laboratory and in the field demonstrate that pure gradient diffusion models are not capable of satisfactorily performing this task. In particular, a strong dependence of the observed Fickian sediment diffusivity, ε_{Fick}, on sediment settling velocity, w_s, has been clearly established by experiments covering a large range of flow situations from laboratory channels, to rivers and even to wave flumes and coastal locations involving oscillatory flows. This positive dependence, ($\uparrow \varepsilon_{Fick}$ for $\uparrow w_s$) implies that the larger, more inert particles are mixed more efficiently by the turbulence than their less heavy counterparts. Coleman [3] obtained an experimental dataset that clearly demonstrated this phenomena, see figure 1. This seemingly strange nature of apparent Fickian sediment diffusivities has typically necessitated the application of ad hoc modifications (β–factors) to this parameter for modelling purposes.

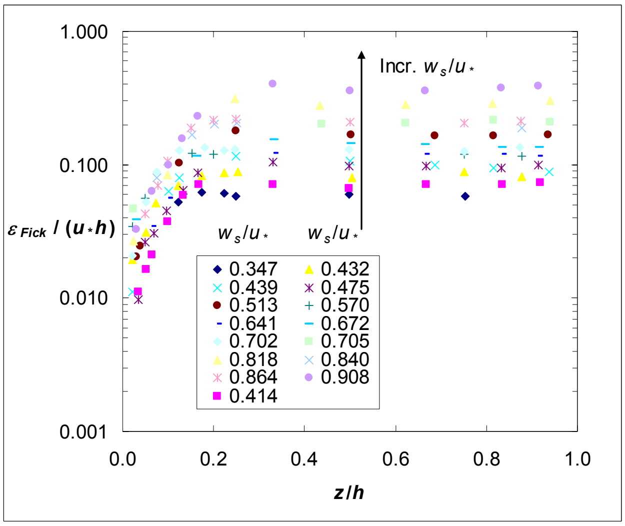


Figure 1. Coleman's [3] dataset clearly showing $\uparrow \varepsilon_{Fick}$ for $\uparrow w_s$.

β is defined as the ratio of the apparent Fickian diffusivity of sediment and of momentum (the eddy–viscosity, ν_t), $\beta = \varepsilon_{Fick} / \nu_t$. This parameter describes the relative turbulent mixing efficiency of momentum compared with sediment in the same flow. β–factors greater than and less than 1 have been observed in turbulent channel and river flows. However, a universally observed feature is the strong dependence in a given series of experiment of β on the ratio, w_s/u_*.

Nielsen [9] explains that the observed behaviour is due to the importance of large–scale convective transport mechanisms in turbulent mixing. However, the assumption implicit in the derivation of the gradient diffusion framework is that mixing occurs on purely infinitesimal scales. Nielsen and Teakle [10] derived a new Finite–Mixing–Length (FML) framework for turbulent diffusion which is capable of explaining the observed trend of increasing ε_{Fick} with increasing w_s/u_*. However, some other features of the data such as $\beta < 1$ for flat–bed laboratory experiments could not be reconciled with the FML model alone. In order to investigate possible improvements to the model, some of the dynamic interactions that may occur between sediment and fluid are also qualitatively considered.

Finite–Mixing–Length Model

Nielsen & Teakle [10] consider in detail the derivation of the FML theory and its application to a number of simple situations. The following section will provide a brief summary of this work. Turbulent mixing is capable of generating a net vertical flux of suspended sediment which can be quantified in terms of the simplified scenario involving the swapping of fluid parcels (including suspended sediment) shown in figure 2.

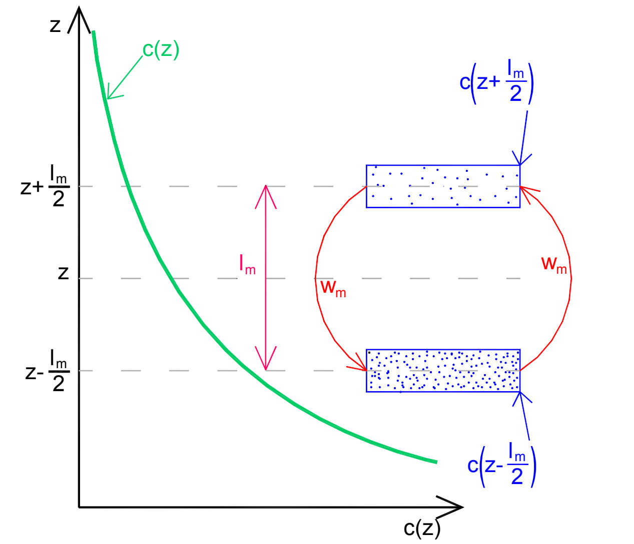


Figure 2. The upward sediment flux due to turbulent mixing can be understood in terms of swapping fluid-sediment-mix parcels over a vertical distance l_m.

The resulting sediment flux density is,

$$q_m = w_m \left[c(z - l_m/2) - c(z + l_m/2) \right] \qquad (1)$$

and by Taylor expansion of $c\left(z \pm l_m/2\right)$,

$$
\begin{aligned}
q_m &= -w_m l_m \frac{\partial c}{\partial z}\left[\sum_{n=1}^{\infty}\left\{\frac{l_m^{(2n-2)}}{(2n-1)!\,2^{(2n-2)}}\frac{\dfrac{\partial^{(2n-1)}c}{\partial z^{(2n-1)}}}{\dfrac{\partial c}{\partial z}}\right\}\right] \\
&= -w_m l_m \frac{\partial c}{\partial z}\left[1 + \frac{l_m^2}{24}\frac{\dfrac{\partial^3 c}{\partial z^3}}{\dfrac{\partial c}{\partial z}} + \ldots\right]
\end{aligned}
\tag{2}
$$

The term in front of the brackets is the familiar gradient diffusion flux. The higher order terms disappear for the case of infinitesimal mixing length, $l_m \to 0$, demonstrating that Fickian diffusion is only a particular limit of the more general turbulent mixing scenario. It is the effect of the higher-order terms for cases of finite l_m which can account for much of the previously irreconcilable behaviour of suspended sediment profiles in turbulent flows.

Apparent Fickian Diffusivities

The sediment continuity equation for a 1D vertical, steady scenario can be written,

$$q_m - w_s c(z) = 0 \tag{3}$$

If the gradient diffusion framework is adopted, the mixing flux is simply assumed to be,

$$q_m = -\varepsilon_{Fick}\frac{\partial c}{\partial z} \tag{4}$$

Using Eqs. (2), (3) and (4) it can be seen that, according to the FML framework, the apparent Fickian diffusivity is,

$$\varepsilon_{Fick} = -\frac{w_s c(z)}{\dfrac{\partial c}{\partial z}} = w_m l_m\left[1 + \frac{l_m^2}{24}\frac{\dfrac{\partial^3 c}{\partial z^3}}{\dfrac{\partial c}{\partial z}} + \ldots\right] \tag{5}$$

Therefore, ε_{Fick}, is not purely a function of the turbulence parameters w_m and l_m, it is also dependent on the higher order terms in Eq. (5). The nature of these higher order terms can be more clearly seen by considering an exponential concentration distribution,

$$c(z) = c(z_0)\,e^{-\frac{(z-z_0)}{L_c}} \tag{6}$$

Substituting this into Eq. (2) gives for the mixing flux,

$$
\begin{aligned}
q_m &= -w_m l_m \frac{\partial c}{\partial z}\left[2\frac{L_c}{l_m}\sinh\left(\frac{l_m}{2L_c}\right)\right] \\
&= -w_m l_m \frac{\partial c}{\partial z}\left[1 + \frac{1}{24}\left(\frac{l_m}{L_c}\right)^2 + \ldots\right]
\end{aligned}
\tag{7}
$$

From Eq. (7) it can be seen that the higher order terms are zero and the turbulent mixing process is correspondingly Fickian only for the limit of $l_m/L_c \to 0$, i.e. when the turbulent mixing length is small compared with the distribution length–scale L_c. Coarser particles (larger w_0) are seen to have larger ε_{Fick} than fine particles due to the fact that they have a relatively smaller L_c (and hence larger higher order terms according to Eq. (7)) in a given turbulent mixing regime.

FML Concentration Distributions

For a couple of simple turbulent mixing scenarios the FML model predicts concentration distributions that are analogous to the corresponding Fickian distributions:

Homogeneous Turbulence; constant w_m and l_m

In this case the concentration profile derived from the pure gradient diffusion theory (combining Eqs. (3) and (4)) gives an exponential distribution, Eq. (6), where the distribution length–scale is given by,

$$L_c = \frac{l_m w_m}{w_s} \tag{8}$$

The FML framework (combining Eqs. (1) and (3)) gives an analogous solution to (6), however in this case the distribution length–scale is,

$$
\begin{aligned}
L_c &= \frac{l_m}{2\sinh^{-1}\left(\dfrac{w_0}{2w_m}\right)} \\
&= \frac{l_m w_m}{w_s}\left(1 + \frac{1}{24}\left(\frac{w_0}{w_m}\right)^2 - \ldots\right)
\end{aligned}
\tag{9}
$$

Figure 3 illustrates this relationship and demonstrates that the Fickian approximation is exact for $w_m/w_s \to 0$, while it underestimates L_c for finite w_s/w_m. Interestingly, van Rijn's [12] empirical β–factor formula shows a similar dependence on w_s/u_* to Eq. (9),

$$\beta = 1 + 2\left(\frac{w_s}{u_*}\right)^2 \tag{10}$$

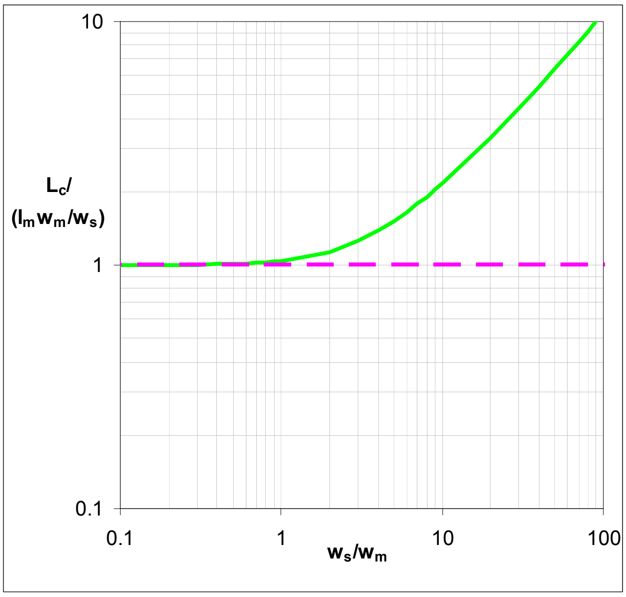


Figure 3. Solid line showing finite-mixing-length model exponential lengthscale, L_c, for homogeneous turbulence compared with the corresponding Fickian prediction (dotted line).

Constant Stress Layer; linear $l_m(z)$ and constant w_m

This turbulence distribution is usually assumed to apply to the "constant–stress–layer" in wall–bounded shear flows, e.g. Pope [11]. In particular Nielsen and Teakle [10] made the general assumption that,

$$l_m(z) = \lambda z \text{, and} \tag{11}$$

$$w_m = \gamma u_* \tag{12}$$

where, $u_* \equiv \sqrt{\tau_0/\rho}$ is the bed, friction–velocity. This is similar to Prandtl's mixing–length hypothesis except that Prandtl made the further assumptions that $\gamma = 1$, $\lambda = \kappa$, and that the mixing process was Fickian in nature.

Nielsen and Teakle [10] sought a unified and consistent turbulent mixing model for both momentum and sediment for the constant stress layer. Firstly, considering the FML model (figure 2) applied to momentum transfer with mixing parameters given by Eqs. (11) and (12), they got,

$$-\rho u_*^2 = \gamma u_* \left[\rho u\left(z - \frac{\lambda}{2} z \right) - \rho u\left(z + \frac{\lambda}{2} z \right) \right] \tag{13}$$

This equation is found to satisfy the usual logarithmic velocity profile,

$$u(z) = \frac{u_*}{\kappa} \ln\left(\frac{z}{z_0} \right) \tag{14}$$

when λ and γ satisfy,

$$\gamma \ln\left(\frac{1+\lambda/2}{1-\lambda/2} \right) = \kappa \tag{15}$$

This equation approaches the Fickian limit $\lambda\ \gamma \rightarrow \kappa$ as $\lambda \rightarrow 0$. Physically $\lambda = 2$ is the upper limit in the mixing scenario of figure 2.

The sediment distribution obtained using the Fickian approximation is a power–function in the case of a constant–stress–layer,

$$c(z) = c(z_0) \left(\frac{z}{z_0} \right)^{-\frac{w_s}{\lambda \gamma u_*}} \tag{16}$$

The FML model also yields a power–function for the sediment distribution,

$$c(z) = c(z_0) \left(\frac{z}{z_0} \right)^{p} \tag{17}$$

where p satisfies,

$$\left(1 - \frac{\lambda}{2} \right)^{p} - \left(1 + \frac{\lambda}{2} \right)^{p} = \frac{w_s}{\gamma u_*} \tag{18}$$

The FML model approaches the Fickian distribution, $p = -w_s / (\lambda \gamma u_*)$, when $w_s/u_* \rightarrow 0$. As $\lambda \rightarrow 0$ the homogeneous FML solution (figure 3) is approached. The strength of the FML mixing enhancement is determined by the magnitude of the mixing length gradient, λ. This raises the issue of how the model parameters λ and γ should be chosen. In the Fickian mixing scenario w_m and l_m only ever appear in combination, i.e. $w_m l_m$, therefore in the Fickian model it does not matter how these terms proportionately contribute to the mixing. However, in the FML model (Eq. (5)) the relative size of the mixing length is seen to determine the strength of the FML effects. Therefore, in this case the model parameters will be chosen to best match the available constant–stress–layer data.

Comparison With Experimental Data

The most convenient way to compare the model with experimental data makes use of the β–factor,

$$\beta = \frac{\varepsilon_{Fick}}{\nu_t} = \frac{-w_s c(z)}{\frac{\partial c}{\partial z}} \Bigg/ \frac{\tau}{\rho \frac{\partial u}{\partial z}} \tag{19}$$

For the constant–stress–layer case (log–function velocity and power–function sediment distribution) this gives,

$$\beta = -\frac{1}{\kappa p} \frac{w_s}{u_*} \tag{20}$$

The comparison of the FML model with data is shown in figure 4, for three different choices of λ. The data points were obtained by only considering the constant–stress–layer portion ($z/h < 0.5$) of the concentration profiles from channel and river flow experiments. In order to maintain a consistent approach between datasets, some of which did not include detailed velocity measurements, it was assumed that κ (in effect the velocity gradient) maintained its clear–water value of 0.4 in all cases, even though this was demonstrated not to necessarily be the case where velocity measurements were available. β is seen to be an increasing function of w_s/u_* in good qualitative agreement with the data. The choice of larger values of λ produces a stronger increase with w_s/u_*. The slope of the data suggests that $\lambda = 1.9$ best describes the increase of β with w_s/u_*. This has an interesting physical interpretation when it is considered that as $\lambda \rightarrow 2$ the mixing scenario shown in figure 2 corresponds to mixing between an upper parcel and a lower parcel that always originates from the vicinity of the bed.

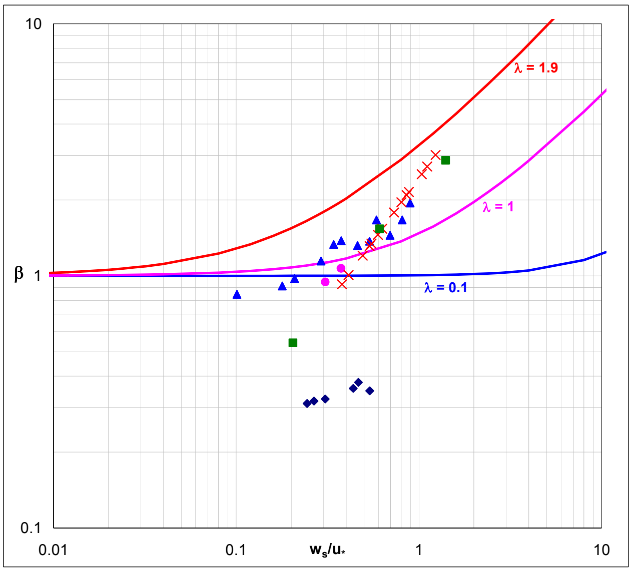


Figure 4 Lines showing theoretical β-values for the constant stress layer model. Experimental β-values from the constant stress layer of channel flows; ♦ Graf & Cellino (no bedforms) [6], ● Graf & Cellino (bedforms) [2], ▲ Anderson (Enoree River) [1], x Coleman [3], □ Coleman [4].

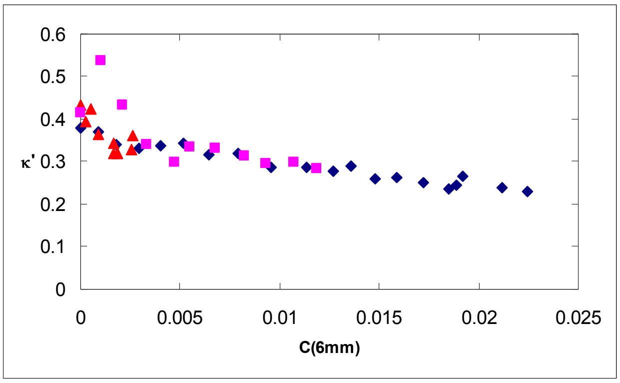
While the FML model predicts increasing β with increasing w_s/u_* in qualitative agreement with the data, the quantitative description shown in figure 4 is not yet convincing. For instance the model predicts $\beta \geq 1$ for all sediment sizes while some of the flat bed experiments indicate $\beta < 1$. The fact that the data does not collapse when plotted in this manner also suggests that processes other than those considered in this analysis are most likely important. Some of these additional processes are briefly considered in the next section.

Dynamic Interaction of Fluid and Sediment

The constant–stress–layer analysis shown in the previous section was based on the assumption that suspended sediment is a passive scalar in a turbulent flow. However this is not usually the case, the presence of suspended sediment is known to change the suspending fluids mean velocity and turbulence profiles from the clear water equivalent. Furthermore, the presence of turbulence may significantly affect the velocity statistics of the suspended sediment due to a process called “selective sampling”. This can lead to the bulk settling velocity of the sediment in turbulence being different from its clear water value, as well as the turbulent fluctuations of the sediment being different from that of the suspending fluid.

Effect of Suspended Sediment on Fluid Velocity

The addition of a suspended load of heavy particles to a steady, uniform, clear–water, channel flow has been shown to increase the near–bed velocity gradient. This is usually explained by turbulence attenuation resulting from the stable stratification induced by the sediment concentration gradients, [7]. This effect implies that the presence of sediment causes momentum to be less efficiently mixed, which should be accounted for in the calculation of β (Eq. (19)). In order to investigate the importance of this stratification effect the dataset of Coleman [4] was considered. Both velocity and sediment concentration measurements were obtained as sediment load was gradually increased in a series of experiments. In order to simply demonstrate any first–order effects, the following analysis assumes that a log–law velocity and power–law concentration profile remain valid. Figure 5a shows how the apparent von Karman constant, κ', varied with suspended sediment concentration (simply a convenient abscissae). Figure 5b compares β obtained assuming κ = 0.4 with that obtained allowing for variation, β'. This shows that, in the case of Coleman's data, $\beta < 1$ in the original analysis can be accounted for by the reduction in velocity gradient. Unfortunately the same explanation does not seem to account for $\beta < 1$ in Graf and Cellino's [6] data.



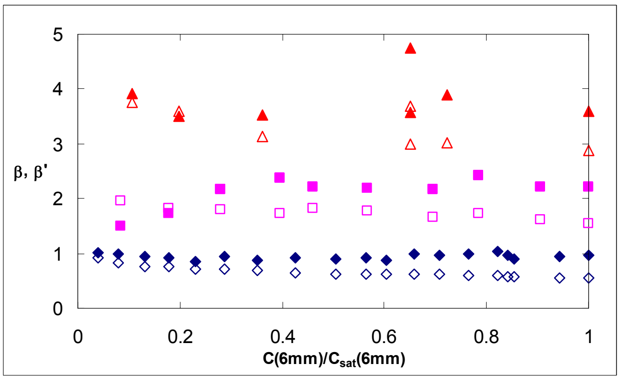


Figure 5a Showing the variation in κ' with increasing sediment load. Data from Coleman (1986) [4]. ◆ $w_s/u_* = 0.2$; ■ $w_s/u_* = 0.6$; ▲ $w_s/u_* = 1.5$. 5b Comparing β (hollow symbols) and β' (solid symbols). Symbols as for 5a.

Sediment Velocity

Most models of suspended sediment and many experiments have assumed that the sediment and fluid have essentially identical velocity statistics. However, an interesting experiment by Muste and Patel [8] used a discriminator LDV to measure separate fluid and suspended sediment velocities in dilute sediment–laden flow. Their results suggest that there is a lag between fluid and sediment mean stream–wise velocities of up to 4%. Their most intriguing finding was that stream–wise turbulence intensities of water and sand were very similar, whereas the vertical sediment turbulence intensities were significantly damped compared with the fluid (by approximately 20%). In the context of the FML model this could be interpreted as a reduction of the sediment mixing velocity, w_m, compared with that of the fluid. This would lead to $\beta < 1$ for small w_s/u_*..

This phenomenon could perhaps be explained by the particles "selectively–sampling" the turbulent fluid velocity field. That is, in a fluid velocity field comprising spatially and temporally coherent turbulent structures, suspended particles with inertia differing from that of a fluid particle have been shown to preferentially migrate due to centrifugal action to certain regions of the flow field [5]. This "preferential concentration" implies that the suspended particles are "selectively sampling" the fluid velocity field and therefore different velocity statistics between the fluid and sediment become possible.

Selective sampling has been shown to change the bulk settling velocity of particles in turbulence from its clear water value. For heavy particles in a strong turbulence field this is expected to result in an increase in bulk settling velocity [13]. This increase in settling velocity due to turbulence would have the same effect on the concentration profiles as a reduction in mixing efficiency and thus would manifest itself as $\beta < 1$.

Conclusions

This paper has demonstrated the application of a Finite–Mixing–Length (FML) theory to the case of sediment suspensions in steady, uniform flows. Unlike the Fickian diffusion framework the FML theory is capable of explaining why β–factors are required to be an increasing function of w_s/u_*. A unified FML description of momentum and sediment mixing in a "constant–stress–layer" was shown to adequately describe the observed trend in the data.

The large scatter in the data for small w_s/u_* and the occurrence of $\beta < 1$ was considered in some further detail. The dynamic interaction of fluid and suspended sediment was shown to significantly alter both the momentum and sediment mixing efficiencies. The potential importance of a "selective–sampling" mechanism was also postulated.

Acknowledgments

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