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A Blob Insertion Technique for the Vortex Blob Method

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Abstract

This paper describes a new vortex blob method for the solution of problems in 2D vortex dynamics. In contrast to traditional vortex blob methods, this approach incorporates the distortion of the blobs, by the solution of two additional ODEs for each vortex blob, giving the evolution of the aspect ratio and orientation for an equivalent elliptical patch of constant vorticity in response to the locally linearised, but time-dependent, strain and vorticity fields. When the aspect ratio of the equivalent ellipse reaches a pre-determined limit, the circular vortex blob is split in two, with the centres of the new blobs aligned along the principal axis of the ellipse, thus providing a consistent approach to vortex blob insertion, so that fine-scale features such as vortex filamentation can be followed in an efficient way.

Introduction

Lagrangian methods have a fundamental advantage over Eulerian or grid-based methods for problems in 2D vortex dynamics, in that the calculations are carried out only where the flow velocities are non-zero. This is particularly so for cases where there are strongly nonlinear interactions, giving rise to small-scale, but localised structures. Thus a vortex patch in a strain field (due to the presence of other vortices nearby e.g.) may deform and eventually produce filaments of vorticity. As time progresses, these filaments may quickly become very long and thin, and are advected by the local velocity field, giving rise to long sinuous streaks of vorticity. For inviscid, or near-inviscid flow, the cross-filamentary scales may be extremely small. Resolving such features is extremely difficult with uniform grids, and even with non-uniform grids there is the requirement of changing the grid very quickly in time as the flow evolves. In response to this, vorticity contour following methods have been developed, starting with the contour dynamics technique of Zabusky et al [13], and evolving to hybrid methods such as the Contour Advective Semi-Lagrangian (CASL) method of Dritschel and Ambaum [4], where the stream-function is inverted on a relatively coarse grid, but the vorticity contours (represented by nodes) are advected using Lagrangian techniques by interpolating the velocity field on to the contour nodes at each time-step. Because of the fact that vorticity contours very quickly lengthen as filamentation takes place, these methods must continously insert new nodes along the vorticity contours, and this growth in the number of nodes can quickly limit the usefulness of such approaches. Accordingly, the techniques of 'contour surgery' [3] have been developed, where contours of vorticity are broken when they become very thin, and equal-valued contours merged when they approach closely.

An alternative Lagrangian approach for such problems is to represent the initial vorticity distribution with point-vortices (each has a $\delta-$distribution of vorticity) and then to determine the motion of each point-vortex in response to all other vortices, so giving the evolution of the vorticity field. This technique was first introduced by Rosenhead [9], in his study of the evolution of a vortex sheet. Difficulties encountered with random motions of such point-vortices are to some extent mitigated by desingularising in some way, i.e. replacing the $\delta-$distribution of vorticity with some smeared non-infinite distribution with a characteristic small scale, the so-called vortex blob approach of Chorin and Bernard [2]. These vortex blobs methods all introduce some numerical dissipation, but are generally accepted as being superior to point-vortex techniques, and we focus on such methods here. In order to resolve fine-scale vorticity features, the vortex blob methods require techniques of artificial blob insertion, that play a role equivalent to the node insertion on contours for contour dynamics techniques. When a single blob is replaced by two others, say, then each of the new blobs will have half the circulation of the old blob. It is clear that any initially circularly symmetric blob should actually distort in response to the local velocity shear. To account for this, Rossi [11] introduced Gaussian elliptical blobs, and determined ODEs for their evolution, to be solved simultaneously with the ODEs describing the motion of the centroids of the blobs. Typically, the aspect ratio of an elliptical blob increases with time, until eventually it must be replaced by a number of elliptical blobs of smaller circulation. The equations for this are non-trivial, however, and in the present work we develop a similar, but simpler scheme. We use fixed size blobs (i.e. no account is taken of viscosity) but for each blob simultaneously follow an equivalent constant vorticity patch of constant area in response to the first order approximation to the local strain and vorticity fields. The two ODEs describing the evolution of the orientation and aspect ratio of the elliptical patch have been obtained by Kida [6], who showed that such a patch remains elliptical for all time. When the aspect ratio reaches some predetermined quantity, we can replace our initial blob with two blobs, each with half the circulation, aligned along the principal axis of the equivalent patch, thus providing a robust and simple technique for introducing blobs in a way consistent with the evolution of the flow-field. Eventually the growth in the number of blobs can be limited by discarding blobs after some suitable number of splittings: again, this introduces numerical dissipation, but this is quantifiable and can be chosen to be consistent with the computational resources available. Note that in general, although not employed here, blob merger should also be allowed, but as our blobs are circular, standard techniques can be used (see Rossi [11]).

This paper reviews the basic equations for the vortex blob approach and then goes on to show how to incorporate the Kida solution for the equivalent elliptical patches. Some preliminary results are provided, and suggestions made for future work.

Mathematical Formulation

Governing Equations

We consider 2-D incompressible, inviscid flow in an unbounded domain, with the flow field governed by the Euler equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = 0 \,, \tag{1}$$

where $\mathbf{u} = (u, v)$ is the fluid velocity and we use rectangular cartesian coordinates (x, y). The corresponding vorticity is thus a scalar, satisfying

$$\frac{D\omega}{Dt} = \frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = 0 \tag{2}$$

and it is convenient to introduce a stream-function ψ with

$$\nabla^2 \psi = \omega. \tag{3}$$

Then $(u, v) = (-\psi_y, \psi_x)$. We define the circulation Γ, associated with any distribution of vorticity $\omega(x, y)$ over a region R by

$$\Gamma = \iint_R \omega(x, y)\; dxdy \,. \tag{4}$$

The 2-D Green's function for Laplace's equation is

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}'|. \tag{5}$$

Therefore, by Green's second identity on our unbounded domain

$$\begin{aligned} \psi(\mathbf{x}) &= \iint \nabla^2 \psi(\mathbf{x}') G(\mathbf{x}, \mathbf{x}')\; dx'dy' \\ &= \frac{1}{2\pi} \iint \omega(x', y') \log |\mathbf{x} - \mathbf{x}'|\; dx'dy' \end{aligned} \tag{6}$$

and hence the fundamental equations for the time evolution of the velocity components are

$$\begin{aligned} \frac{dx}{dt} &\equiv u = -\frac{\partial \psi}{\partial y} = -\iint \omega(x', y') \frac{\partial G(\mathbf{x} - \mathbf{x}')}{\partial y}\; dx'dy' \\ &= \frac{1}{2\pi} \iint \omega(x', y')\; \frac{(y' - y)}{(x - x')^2 + (y - y')^2}\; dx'dy' \end{aligned} \tag{7}$$

and similarly

$$\frac{dy}{dt} \equiv v = \frac{1}{2\pi} \iint \omega(x', y')\; \frac{(x - x')}{(x - x')^2 + (y - y')^2}\; dx'dy' \,. \tag{8}$$

All Lagrangian methods solve equation (1) by determining u and v at time t in some way, e.g. from expressions like equations (7) and (8) (point-vortex [9] and vortex-blob [2] methods), by evaluating a line integral over the Green's function (contour dynamics [13] methods) or by inverting equation (3) numerically to find a stream function (the CASL method [4]; [8]). This velocity field is then used to advect the vorticity out to the next timestep.

Point-Vortex and Vortex Blob Methods

Consider the evolution of a patch of vorticity, non-zero on some region D. If we represent this as

$$\omega(\mathbf{x}) = \sum_{j=1}^{N} \Gamma_j \delta(\mathbf{x} - \mathbf{x}_j) \tag{9}$$

then we have the so-called point-vortex method. Here we represent $\omega(\mathbf{x})$ by the linear superposition of N point vortices, located at positions $\mathbf{x}_j,\ j = 1, \ldots, N$ and with associated circulation

$$\Gamma_j = \iint \Gamma_j \delta(\mathbf{x} - \mathbf{x}_j) d\mathbf{x} \,. \tag{10}$$

Substitution of equation (9) in equations (7) and (8) gives explicit expressions for dx/dt and dy/dt for each of the point vortices and then setting $\mathbf{x} = \mathbf{x}_i$ gives the velocity with which point vortex i is advected by the flow field corresponding to all the other vortices. Use of a standard ODE solving routine (4th-order Runge-Kutta is used here) provides an algorithm for the evolution of the original patch of vorticity. This is the method first used by Rosenhead [9].

There are two fundamental issues/difficulties associated with the point-vortex method. One is the degree to which a general distribution of vorticity can be accurately represented by the point vortices. This can be addressed by using more vortices, but of course that increases the cost of the calculation. Perhaps more significantly, the interaction of four or more vortices gives rise to intrinsically chaotic motions [1], so that in general the long-time behaviour of such models tends to be unreliable.

One way to deal with these problems is to 'desingularise' the point vortices, leading to so-called vortex blobs. This idea goes back to Chorin and Bernard [2], but the explicit model we use here is due to Krasny [7] (his $\delta-$equation method):

$$\omega(\mathbf{x}) = \frac{1}{\pi} \sum_{j=1}^{N} \Gamma_j\; \frac{\delta^2}{((x - x_j)^2 + (y - y_j)^2 + \delta^2)^2} \,. \tag{11}$$

There is no longer a singularity at $\mathbf{x} = \mathbf{x}'$ but the $\delta-$function representation is regained in the limit when $\delta \to 0$. Here δ, where ω has dropped to one quarter of its maximum (central) value, can be thought of as the radius of the blob. The smearing of the point vortex introduced in this way acts to reduce the tendency to chaotic motion, essentially because there is an effective dissipation introduced. Even with this desingularisation, however, there is a tendency for individual vortex blobs to move chaotically, particularly in cases where filamentation takes place and the separation between blobs becomes large. This problem can be overcome to some extent by splitting blobs, or by introducing new blobs and renormalising the circulation.

Blob-Splitting Determined by the Local Strain Field

The contribution of the present work is a method that allows vortex blob insertion in a dynamically consistent fashion. The approach is motivated by the work of Rossi ([11], [12]), who introduced the idea of using elliptical Gaussian basis functions (or blobs) that deform according to the prevailing velocity field. Once the blobs become significantly elongated, they can be split and replaced by a number of smaller elliptical blobs. In addition, diffusive spreading due to viscosity (not treated here) can be included. However, the equations describing the deformation of the elliptical blobs are non-trivial, so that there is an associated increase in computational load. On the other hand, this is more than compensated for by the increased spatial accuracy.

The simpler idea invoked here is to work with non-deforming circular blobs, but to keep track of their potential deformation under the prevailing strain field due to all the other blobs. To implement this, it is necessary to integrate two ODEs for each blob. These ODEs determine the axis-ratio (p) and orientation (θ) as a function of time for the corresponding 'equivalent elliptical patch' of uniform vorticity, initially circular with radius δ and with the same circulation, in response to the local linearisation of the velocity field induced by all the other vortex blobs. This is possible because exact ODEs for this problem have been determined by Kida [6] for this situation. Two new blobs, with half the circulation, and with centres separated by distance 2δ and aligned along the major axis of the ellipse equivalent to the vortex blob, are used to replace a vortex blob when the effective aspect ratio for a blob p_i exceeds the critical value p_{crit}. In this way we can resolve fine-scale features of the flow, such as filamentation, by introducing new blobs as required by the local nature of the velocity field. As this procedure will eventually lead to an exponential growth in the number of blobs, some way of limiting the growth is required. The natural technique is to discard blobs after a certain number of splittings (say 9 or 10) when their individual contribution to the circulation is reduced to 1/512 or 1/1024 respectively of that of an original blob. Note that although the constant vorticity elliptical patch is not exactly equivalent to the corresponding vortex blob, there is no additional error introduced, as we are only using this representation as an aid to deciding when and how to split the vortex blob.

Implementation of the Kida Solution for Elliptical Patches

For any given vortex blob we expand the external velocity field due to the other blobs to first order as:

$$\mathbf{u} - \mathbf{u}_B \approx A\mathbf{x} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}_B \mathbf{x} \tag{12}$$

where $\mathbf{u}_B$ and all the shear components are evaluated at the centre of the blob and where the coordinate origin here is at the centre of the blob. This corresponds to the external shear field considered by Kida [6], but, of course, here these quantitities are not fixed in time but vary as the vorticity field and the corresponding velocity field evolve.

From continuity $u_x = -v_y$ and then

$$\begin{aligned} \begin{bmatrix} u_x & u_y \\ v_x & -u_x \end{bmatrix} &= \begin{bmatrix} u_x & 0 \\ 0 & -u_x \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & (u_y+v_x) \\ (u_y+v_x) & 0 \end{bmatrix} \\ &+ \frac{1}{2}\begin{bmatrix} 0 & -(v_x-u_y) \\ (v_x-u_y) & 0 \end{bmatrix} \\ &= \begin{bmatrix} e & 0 \\ 0 & -e \end{bmatrix} + \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\gamma \\ \gamma & 0 \end{bmatrix} \end{aligned} \tag{13}$$

where $\gamma = (v_x - u_y)/2$ is half the local vorticity, e is the strain rate and $\beta = (u_y + v_x)/2$. By rotating coordinates to align with the principal strains (following Kida [6]) we find

$$P(\mathbf{u} - \mathbf{u}_B) = D\bar{\mathbf{x}} + \Lambda\bar{\mathbf{x}} \tag{14}$$

where

$$D = \begin{bmatrix} e_{\text{eff}} & 0 \\ 0 & -e_{\text{eff}} \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} 0 & -\gamma \\ \gamma & 0 \end{bmatrix}, \tag{15}$$

and the coordinates $\bar{\mathbf{x}}$ are rotated by an angle ϕ from the original coordinates $\mathbf{x}$. In fact

$$D = PAP^T \tag{16}$$

where D is the diagonal matrix of eigenvalues of the matrix A and

$$P = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \tag{17}$$

with the columns of P the (suitably normalized) orthonormal eigenvectors of A. Here the eigenvalues

$$e_{\text{eff}} = \pm\sqrt{e^2+\beta^2} = \pm\sqrt{u_x^2 + \left(\frac{u_y+v_x}{2}\right)^2}\,. \tag{18}$$

It turns out that the transformation of velocities implied by equation (14) can be ignored as the only inputs to the Kida model are e_{eff} and γ (see below).

The equations for the evolution of the i th elliptical patch are

$$\begin{aligned} \dot{p}_i &= 2e_{\text{eff}}p_i\cos 2\theta_i \\ \dot{\theta}_i &= -e_{\text{eff}}\left(\frac{p_i^2+1}{p_i^2-1}\right)\sin 2\theta_i + \frac{\Gamma_i}{2\pi}\frac{p_i}{(p_i+1)^2} \\ &+ \gamma_i\,, \end{aligned} \tag{19}$$

with $p_i = a_i/b_i$ the ratio of the lengths of the major and minor semi-axes of the ellipse. These are Kida's equations (3.2) and (3.3) with minor changes of notation. Note that θ_i is the angle relative to the principal axes of strain, so the orientation $\theta_{i,\text{ellipse}}$ of the principal axis of the elliptical patch relative to the original axes is given by

$$\theta_{i,\text{ellipse}} = \theta_i + \phi_i \tag{20}$$

where ϕ_i is obtained for each vortex blob from the eigenvector corresponding to the eigenvalue $e_{i,\text{eff}}$, i.e.

$$\phi_i = \cos^{-1}\left(\frac{\beta_i}{\sqrt{(e_{i,\text{eff}}-e_i)^2+\beta_i^2}}\right) \tag{21}$$

and clearly all angles are functions of time. Finally we note that the initial value of θ_i can be chosen to be zero, corresponding to an alignment with the local principal axes of strain, as the blobs are all circular.

A Numerical Example

As a test case we consider the interaction of two circular patches of uniform unit vorticity (Rankine vortices) of radius $R = 1$, with their centres separated by a relatively short distance $3R$, so that merger is expected. Each vortex is represented initally by 7 rings of vortex blobs and one central blob, all with equal strength and with 'radius' $\delta = 1/15$, so covering the vortex. As each blob represents an equal area of the patch, there are $8n$ blobs in ring n, giving 225 blobs initially for each vortex, following Hume [5]. A fourth-order variable time-step Runge-Kutta routine is used (**ode45.m** in MATLAB), and 're-blobbing' carried out every 0.5 time units. The area of blobs in the figures is shown as proportional to the associated circulation, thus giving a visual representation of the relative dynamic significance of each blob.

In figure 1 the results at times $t = 10, 20$ and 30 are given, with $p_{\text{crit}} = 5$. Frames (a), (b) and (c) show the blobs, with area proportional to their associated circulation. In frames (d), (e) and (f), the corresponding contour plots of vorticity are given, with the colour bar giving vorticity magnitude. (To smooth the contours of vorticity, we have used a value of $\delta = 0.2$ when *evaluating* the vorticity from the blob distribution using equation (11)). Also

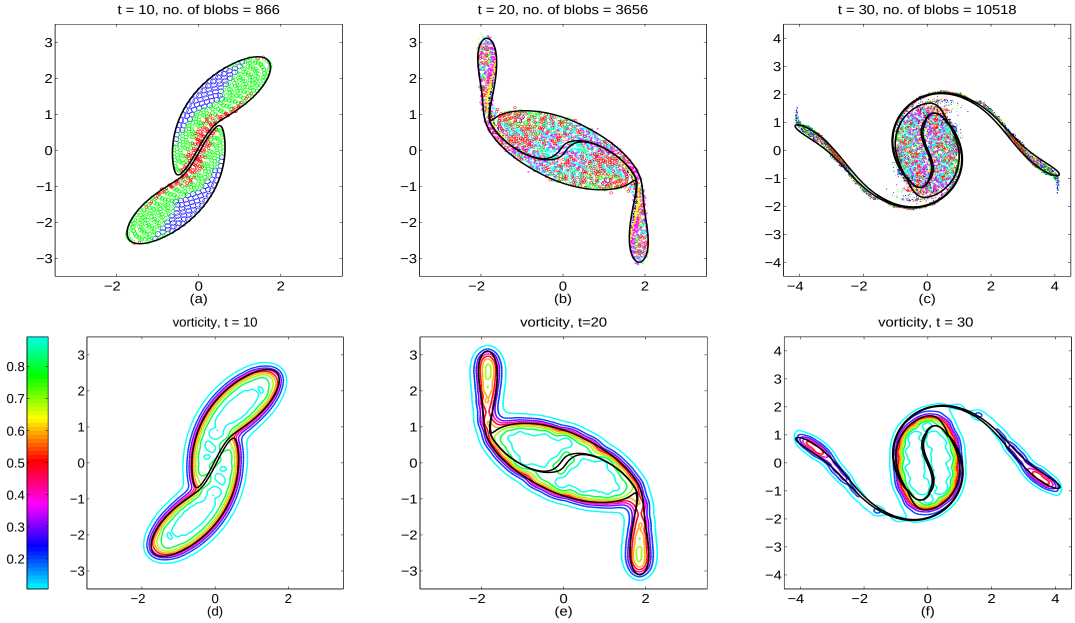

Figure 1: The behaviour at times $t = 10, 20$ and 30 for the merger of two Rankine vortices, with unit radius and centres separated by three radii. Panels (a), (b) and (c) show the vortex blobs, colour coded and with areas proportional to their corresponding circulations, while panels (d), (e) and (f) show the corresponding contour plots of vorticity. In all cases the corresponding contour dynamics calculation (Dritschel [3]) is shown as a solid black curve.

shown are the contours corresponding to the vorticity jump from zero to unity in a planar high resolution contour dynamics calculation, with minimal contour surgery (Dritschel [3]). Very good agreement is achieved, with the filaments of vorticity well resolved, but at this stage the vortex blob code is not very efficient.

Conclusions

This paper has provided a novel technique for adjusting the resolution of vortex blob methods by introducing new blobs as required, according to the time evolution of an equivalent elliptical patch for each circular blob, thus resolving fine-scale features of the vorticity. Because only circular blobs are used, it is expected that this technique can be quantified accurately, and that simple models for the transfer of enstrophy to high wavenumbers, and its associated final dissipation at the smallest scales resolved, may be possible.

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