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MODELLING SCALE-DEPENDENT RADIAL TWO-PHASE FLOW OF LIQUID AND GAS IN UNSATURATED POROUS MEDIA

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ABSTRACT

In this paper we investigate scale-dependent radial two-phase flow of liquid and gas in unsaturated porous media. First we present an equation of two-phase flow of liquid and gas in porous media in radial coordinates (ETPR) based on McWhorter's equation in Cartesian coordinates. Then we present three analytical solutions for the ETPR: [1] an exact solution of ETPR in three dimensions for radial two-phase flow into a spherical domain subject to an instantaneous point source input; [2] an exact solution of ETPR in two dimensions for two-phase outflow from a well subject to an instantaneous line source input; and [3] an equation of the two-phase wetting front movement in three dimensions is presented based on a coordinate transform, yielding a moving boundary problem in two-phase flow in porous media, and an analytical solution is given which can be used to infer a fractal parameter of the porous media, m. These analytical solutions can aid in the verification of numerical solutions, and explicitly illustrate the functional relationships between the variables and parameters affecting two-phase flow in unsaturated porous media.

NOMENCLATURE

θ volumetric content of the two-phase fluids
θ_i initial value of θ
θ_r value of θ at residual saturation
θ_s value of θ at saturation
$\varsigma(t_\sigma)$ moving point on the wetting front
ξ moving coordinate
ρ_w density of wetting phase, $[ML^{-3}]$
ρ_a density of non-wetting phases, $[ML^{-3}]$
C relative density given by Eq. (3)
C_1 constant of integration
C_2 constant of integration.
D diffusivity
D_w one-phase diffusivity (wetting phase only), $[L^2T^{-1}]$
D_2 constant with the dimensions, $[L]^{2+m}[T]^{-1}$
D_3 constant with the dimensions, $[L]^{2(1+m)}[T]^{-1}$
$erfc$ the complementary error function
D_0, α constants in $D \sim S$ relationship
$D(r)$ scale-dependent two-phase diffusivity, $[L^2T^{-1}]$
f_w the fractional flow function: $f_w = q_w/(q_w + q_n)$
F_{w1} the alternative fractional flow function
j number of dimensions
$K(S)$ two-phase unsaturated hydraulic conductivity, $[LT^{-1}]$
K_w one-phase unsaturated hydraulic conductivity (wetting phase only), $[LT^{-1}]$
m a parameter in the group transformation representing spatial fractal dimension
M_2 physical instantaneous line source strength, $[L^2]$
$\bar{M}_2$ dimensionless instantaneous line source strength
M_3 physical instantaneous point source strength, $[L^3]$
$\bar{M}_3$ dimensionless instantaneous point source strength
q_n flux of non-wetting phases, $[L^3T^{-1}]$
q_w flux of wetting phase, $[L^3T^{-1}]$
r radial coordinate, $[L]$
S the saturation given as: $S = (\theta - \theta_r)/(\theta_s - \theta_r)$
$\bar{S}$ normalised saturation
t time, $[T]$
t_σ transformed time, $[LT^{-4}]$
U velocity of the wetting front, $[LT^{-1}]$
V_j flux or velocity of injected fluid, $[L^j][T^{-1}]$
V_2 flux in two dimensions, $[L^2][T^{-1}]$
V_3 flux in three dimensions, $[L^3][T^{-1}]$

INTRODUCTION

In this paper, we are concerned with two-phase flow of liquid and gas in unsaturated porous media, a process occurring in many natural and artificial environments. Typical examples of two-phase flow are oil-gas displacement in natural petroleum reservoirs (Barenblatt et al., 1990, p. 230), material transport problems in the environment (Morel-Seytoux, 1973; Miller, et al., 1998), water and air flow in unsaturated soils (Green and Ampt, 1911), water and air flow in soils during conventional flood irrigation (Dixon and Linden, 1972), and water and air (or oxygen) movement in soils during aerated subsurface drip irrigation (Bhattarai et al., 2004).

The reader is referred to Morel-Seytoux (1973), Barenblatt et al. (1990) and Miller et al. (1998) for reviews on studies of generic two-phase and multiphase flow, and to Bhattarai et al. (2005) for a specific review on issues of two-phase flow in aerated drip irrigation.

In hydrology and soil physics, the first significant quantitative analysis of two-phase flow of water and air was due to Green and Ampt (1911), who presented the first infiltration equation, based on the Poiseuille's capillary law, which is earlier than such major methods as Buckley-Leverett (1942) and Rapoport-Leas (1953) equations used in modelling multiphase flow. Later, few researchers in hydrology and soil physics pursued this issue (Powers, 1934; Baver, 1937; Lewis and Powers, 1939; Elrick, 1961). The effect of airflow on water movement in irrigated and non-irrigated soils has been largely ignored in subsequent studies based on the assumption that the effect of the air flow is negligible. Following decades of neglect of the water movement and associated soil air effects, H. Morel-Seytoux and his co-workers reactivated the investigation of two-phase flow of water and air in porous media, and G.C. Sander and his colleagues continued the quantitative investigations into the two-phase flow. The representative works of these two groups and their associates include Brustkern and Morel-Seytoux (1970), McWhorter (1971), Morel-Seytoux (1973; 1983), Parlange and Hill (1979), Parlange et al. (1982), Sander and Parlange (1984), Sander et al. (1984; 1988a; 1988b; 1988c; 1993; 2005), McWhorter and Sunada (1990), and Weeks et al. (1994; 2003). Other investigations include Wilson and Luthin (1963), Vachaud et al. (1973; 1974), Philip and van Duijn (1999) to list a few.

This paper is concerned with the radial two-phase flow in unsaturated porous media. We first present a framework for two-phase flow in radial coordinates, then we develop solutions to the radial two-phase flow equation in three dimensions when the input is point source, and in two dimensions when the input is a line source. Furthermore, we derive an equation of the wetting front movement using a coordinate transformation to yield a two-phase moving boundary problem.

The issues addressed in this paper are important to flow from point and line sources encountered in engineering and hydraulics.

MODEL DESCRIPTION

The partial differential equation governing the movement of two immiscible incompressible fluids is given by several authors (McWhorter, 1971, p. 8; Morel-Seytoux, 1973, p. 144; McWhorter and Sunada, 1990, p. 400). Sander et al. (1988c) rewrote McWhorter's formulation in the following form with notation more familiar in hydrology and soil physics literature,

$$\frac{\partial S}{\partial t} = \frac{\partial}{\partial z}\left[D(S)\frac{\partial S}{\partial z}\right] - \frac{\partial}{\partial z}\left[K(S) + V(t)f_w(S)\right] \quad (1)$$

where

$$V(t) = q_w + q_n \quad (2)$$

$$C = 1 - \rho_n / \rho_w \quad (3)$$

$$D(S) = D_w(1 - f_w) \quad (4)$$

$$K(S) = K_w(1 - f_w) \quad (5)$$

McWhorter (1971), Sander and Parlange (1984), Sander et al. (1984), and Sander et al. (1988a, 1988b, 1988c) extensively investigated Eq. (1) for concurrent flow of water and air.

The above formulation was developed for one-dimensional flow only, and can be extended to three dimensions (Su and Midmore, 2005),

$$\frac{\partial S}{\partial t} = \nabla \bullet \left[D(S)\nabla S - V(t)f_w(S)\right] - \frac{\partial K(S)}{\partial z} \quad (6)$$

Equation (6) is in Cartesian coordinates, and in the present analysis, we investigate the counterpart of Eq. (6) in radial coordinates in which the gravity term can be removed to yield

$$\frac{\partial S}{\partial t} = \frac{1}{r^{j-1}}\frac{\partial}{\partial r}\left[D(S)r^{j-1}\frac{\partial S}{\partial r} - V(t)f_w(S)\right] \quad (7)$$

Equation (7) is a standard Fokker-Planck equation (FPE) in radial coordinates and its mathematical structure is identical to the one given by Carslaw and Jaeger, (1959) and Philip (1994) except that the present form is for two-phase flow. McWhorter and Sunada (1990, p. 403), Weeks et al. (1994) and Sander et al. (2005) investigated a form of Eq. (7) in two dimensions, and Weeks et al. (2003) presented first integral and similarity solutions to Eq. (7) in different dimensions.

RESULTS

We present two types of solutions here for Eq. (7):

[1] two solutions subject to instantaneous source input for a scale-independent diffusivity by extending Philip's (1994) findings to three- and two-dimensional flows, and

[2] the scale-dependent equation of wetting front movement (WFM) and a solution of WFM subject to the boundary condition of the first kind.

The scale-dependent equation of radial two-phase flow in unsaturated porous media

It is obvious that different forms of $V(t)$ and $f_w(S)$ in Eq. (7) will lead to different solutions. In the following, we set $V(t) = V_j$ to be constant in each dimension.

The above formulation for the fractional flow function neglected the effect of capillary gradients and gravitational force. When the effect of capillary gradients and gravitational force are retained, McWhorter (1971, p. 10) shows that F_{w1}, instead of f_w, should be used, and in a horizontal case, it is given as

$$F_{w1} = f_w - \frac{D}{V}\frac{\partial S}{\partial x} \quad (8)$$

When $D(S)$ is a power function,

$$D(S) = D_0 S^{\alpha} \quad (9)$$

Weeks et al. (2003, p. 5) showed that the fractional flow function for *n*-dimensional flow is given as

$$f(S) = S^{n\alpha/2+1} \quad (10)$$

In order to analyse the effect of the strong convective two-phase flow due to a large injection rate where the diffusion coefficient can be regarded as a constant, we have $\alpha = 0$ in Eqs. (9) and (10), then we arrive at

$$f_w = S \quad (11)$$

These conditions can be justified in practice by the fact that a constant rate of flow is present. Of course, a functional *D* can be used which does not contravene the illustrative analyses presented in this paper.

McWhorter's (1971, p.21, Fig. 9) data supports a nearly linear relationship between F_{w1} and S when the capillary gradients are retained, i.e.,

$$F_{w1} = S \tag{12}$$

With f_w being replaced by $F_{m1} = S$ in Eq. (4), and with Fujita's (1952, p. 757) one-phase diffusivity given by

$$D_w = \frac{D}{1-S} \tag{13}$$

Eq. (7) is written

$$\frac{\partial S}{\partial t} = \frac{1}{r^{j-1}}\frac{\partial}{\partial r}\left(Dr^{j-1}\frac{\partial S}{\partial r}\right) - \frac{V_j}{r^{j-1}}\frac{\partial S}{\partial r} \tag{14}$$

In the following sections, we analyse Eq. (14) for three- and two-dimensional problems, and also derive an equation of the wetting front movement.

Scale-independent solutions of two-phase flow equation in radial coordinates

We present solutions of Eq. (14) based on Philips' (1994) studies of one-phase problem identical to Eq. (14).

Following our recent development (Su et al., 2005) in extending the diffusion coefficient to account for the scale-dependence, we modify the diffusion coefficient in Eq. (14) as

$$\frac{\partial S}{\partial t} = \frac{1}{r^{j-1}}\frac{\partial}{\partial r}\left(D(r)r^{j-1}\frac{\partial S}{\partial r}\right) - \frac{V_3}{r^{j-1}}\frac{\partial S}{\partial r} \tag{15}$$

Three-dimensional instantaneous point source solution

The three-dimensional problem defined by Eq. (14) corresponds to the radial flow from a point source as shown in Figure 1.

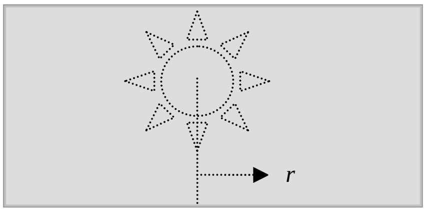

Figure 1. Schematic illustration of radial flow in 3D

With the following group transformations introduced by Philip (1994, Eq. (35), p. 3547),

$$\sigma_* = \frac{1}{3}r^3 \tag{16}$$

$$D(r) = \frac{D_3}{3^{4/3}}\sigma_*^{-2m/3} \tag{17}$$

$$\sigma = \sigma_*\left(\frac{V_3}{D_3}\right)^{3/(2m-1)} \tag{18}$$

$$t_\sigma = t\left(\frac{D_3^3}{V_3^{2(1+m)}}\right)^{1/(2m-1)} \tag{19}$$

Eq. (15) is transformed to

$$\frac{\partial S}{\partial t_\sigma} = \frac{\partial}{\partial \sigma}\left(\sigma^{2(2-m)/3}\frac{\partial S}{\partial \sigma}\right) - \frac{\partial S}{\partial \sigma} \tag{20}$$

Equation (20) is a scale-dependent dispersion-convection equation, and is identical to the one defined by Philip (1994, p. 3547) for one-phase flow. We have physical instantaneous point source strength, M_3, and its normalised instantaneous point source strength, $\bar{M}_3$, defined by Philip (1994, p. 3547) as

$$\bar{M}_3 = M_3\left(\frac{V}{D_3}\right)^{3/(2m-1)} \tag{21}$$

The normalised saturation, $\bar{S}$, is defined as

$$\bar{S} = S/\bar{M}_3 \tag{22}$$

The conditions are defined as

$$t_\sigma = 0, \quad 0 \le \sigma \le \infty, \quad S = \delta(\sigma, t_a) \tag{23}$$

$$t_\sigma > 0, \ \sigma = 0, \ S = \lim_{\sigma \to 0}\left(\sigma^{2(2-m)/3}\frac{\partial S}{\partial \sigma} - S\right) \tag{24}$$

With the above conditions, the solution for $m = 2$ has been given by Philip (1994, p. 3547 – 3548), after restoring the original variables using Eqs. (16) to (19), as

$$S = \frac{V_3^2 M_3}{D_3^{3/2}\sqrt{\pi t}}\exp\left\{-\frac{V_3^2}{36D_3 t}\left[\left(\frac{V_3 r}{D_3}\right)^3 - \frac{3D_3 t}{V_3^2}\right]^2\right\} - \frac{1}{2}\exp\left[\frac{1}{3}\left(\frac{V_3 r}{D_3}\right)^3\right] erfc\left\{\frac{V_3}{6\sqrt{(D_3 t)}}\left[\left(\frac{V_3 r}{D_3}\right)^3 + \frac{3D_3 t}{V_3^2}\right]\right\} \tag{25}$$

which describes the variation of saturation subject to an instantaneous input in a 3D radial space.

Figure 2 illustrates Eq. (25) for the development of saturation profiles as a function following an instantaneous point input.

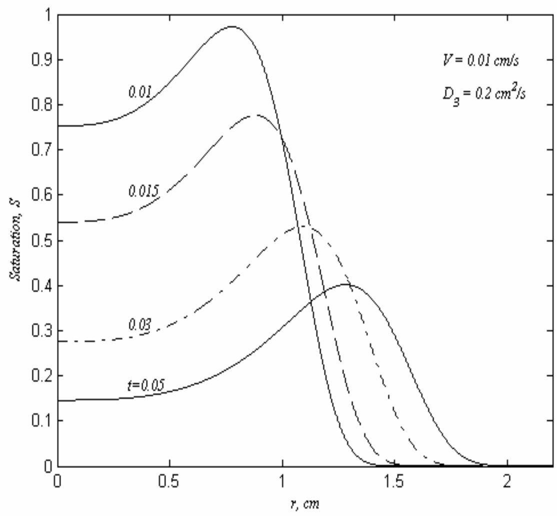

Figure 2. Saturation profiles for an instantaneous point source input in three dimensions

Two-dimensional solution subject to an instantaneous line source

The two-dimensional problem defined in Eq. (15) corresponds to the physical layout shown in Figure 3. The example of this physical layout is flow in wells.

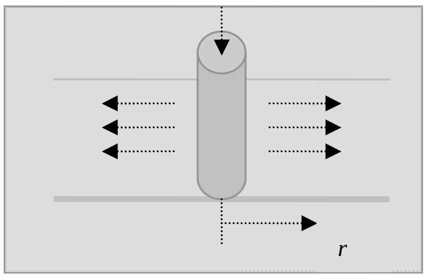

Figure 3. Schematic illustration of radial flow in 2D (such as flow from a well)

For the two-dimensional problem, the transformations given by Philip (1994, p. 3545 – 3547, Eq. (6)), for $m = 2$,

$$D = \frac{1}{2} D_2 r^{-1} \quad (26)$$

$$\sigma = \frac{V_2 r}{2D_2} \quad (27)$$

$$t_\sigma = \frac{V_2^2}{D_2} t \quad (28)$$

reduce Eq. (14) to

$$\frac{\partial S}{\partial t_\sigma} = \frac{\partial}{\partial \sigma}\left(\frac{\partial S}{\partial \sigma}\right) - \frac{\partial S}{\partial \sigma} \qquad m = 2 \quad (29)$$

The physical instantaneous line source strength in two dimensions, M_2, and its normalised strength, $\bar{M}_2$, are defined (Philip, 1994, p. 3547) as

$$\bar{M}_2 = \frac{M_2 V_2}{D_2} \quad (30)$$

The normalised saturation, $\bar{S}$, is defined as

$$\bar{S} = S / \bar{M}_2 \quad (31)$$

The initial and boundary conditions are defined as

$$t = 0\,,\ 0 \le \sigma \le \infty\,,\ S = \delta(\sigma) \quad (32)$$

$$t > 0\,,\ \ \sigma = 0\,,\ \ S = \frac{\partial S}{\partial \sigma} \quad (33)$$

The above problem has a solution of the form (*Philip*, 1994, Eq. (17)), after restoring the original variables,

$$S = \frac{M_2}{\sqrt{\pi D_2 t}} \exp\left[-\frac{\left(V_2 r^2 - 2V_2^2 t\right)^2}{16 D_2 V_2^2 t}\right] - \frac{1}{2}\exp\left(\frac{V_2 r^2}{2D_2}\right) erfc\left[\frac{\left(V_2 r\right)^2 + 2V_2^2 t}{4V_2\left(D_2 t\right)^{1/2}}\right] \quad (34)$$

Equation (34) is plotted in Figure 4 to illustrate the variability of saturation subject to a set of hypothetic parameters.

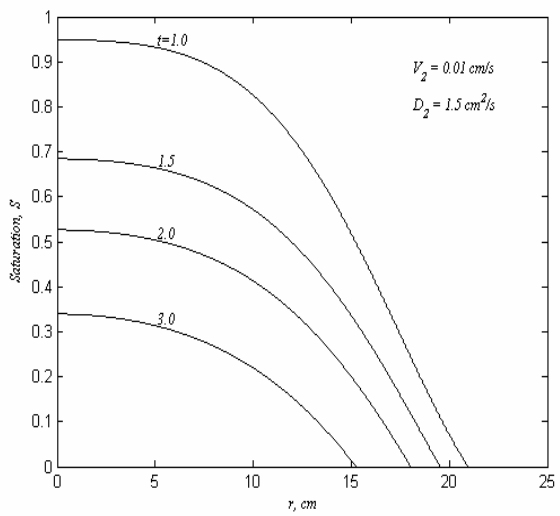

Figure 4. Saturation profiles corresponding to an instantaneous line source input in two dimensions

Equation of scale-dependent wetting front movement of two-phase flow in 3D radial coordinates and its solution

In this case, we are only interested in the movement of the wetting front in a semi-infinite media. We introduce a moving coordinate with a new variable, ξ, (Morel-Seytoux, 1973, p. 136; Broadbridge, 1990, p. 2436),

$$\xi = \sigma - \varsigma(t_\sigma) \quad (35)$$

which changes the origin of the coordinate at the moving surface. With Eq. (35), Eq. (20) is transformed to

$$\frac{\partial S}{\partial t_\sigma} + \frac{\partial S}{\partial \sigma}\left(1 - \frac{\partial \varsigma(t_\sigma)}{\partial t_\sigma}\right) - \frac{\partial}{\partial \sigma}\left(\sigma^{2(2-m)/3}\frac{\partial S}{\partial \sigma}\right) = 0 \quad (36)$$

In the wetting front, the fluid profile propagates by translation, and an observer travelling with the front does not experience a change of the profile with time (Morel-Seytoux, 1973, p. 136), i.e., we have $\partial S / \partial t_\sigma = 0$ in Eq. (36), then we arrive at,

$$\frac{\partial S}{\partial \xi}\left(1 - \frac{d\varsigma(t_\sigma)}{dt_\sigma}\right) - \frac{\partial}{\partial \xi}\left(\sigma^{2(2-m)/3}\frac{\partial S}{\partial \xi}\right) = 0 \quad (37)$$

As the wetting front velocity depends on the fluid saturation, when the initial saturation is uniform we will expect a uniform velocity, U, then we have

$$\varsigma(t_\sigma) = U t_\sigma \quad (38)$$

or

$$\frac{d\varsigma(t_\sigma)}{dt_\sigma} = U \quad (39)$$

which is equivalent to replace Eq. (35) with (Chen, 1988, p. 696),

$$\xi = \sigma - U t_\sigma \quad (40)$$

Equations (38) and (40) are used in Eq. (37) to give

$$\frac{dS}{d\xi}\left(1 - U\right) - \frac{d}{d\xi}\left[\left(\xi + V t_\sigma\right)^{2(2-m)/3}\frac{dS}{d\xi}\right] = 0 \quad (41)$$

We solve Eq. (41) with the boundary conditions for a semi-infinite domain,

$$S = 1 \qquad \xi = 0 \quad (42)$$

$$S \to 0 \quad \frac{dS}{d\xi} \to 0 \qquad \xi \to \infty \quad (43)$$

Equation (42) implies that at the wetting front, the media is saturated, which is a typical assumption devised by Green and Ampt (1911) and widely used in hydrology and soil physics.

As we analyse the wetting front movement, following Morel-Seytoux (1973, p. 137), we solve Eq. (41) for

$$\frac{\partial S}{\partial \xi} = \frac{\partial S}{\partial \sigma} \quad (44)$$

which is equivalent to $\varsigma(t_\sigma) = 0$ at the downstream end. And for small values of V and/or t_σ as we will see in the following example, $\varsigma(t_\sigma) \approx 0$, then Eq. (41) simplifies to an ordinary differential equation in ξ

$$\frac{dS}{d\xi}\left(1 - U\right) - \frac{d}{d\xi}\left[\xi^{2(2-m)/3}\frac{dS}{d\xi}\right] = 0 \quad (45)$$

Equation (45) is solved subject to the boundary conditions in Eqs. (42) and (43) to give

$$\xi = \left[\frac{(2m-1)\ln(1-S)}{3(1-U)}\right]^{3/(2m-1)} \quad (46)$$

or which can be rearranged to give

$$S = 1 - \exp\left[\frac{3(1-U)}{(2m-1)}\xi^{(2m-1)/3}\right] \quad (47)$$

The saturation-position relationship in Eq. (46) or (47) offers a very simple and convenient method for determining the value of m given measured values of S, ξ and U.

The original variables and parameters can be restored in Eq. (47) using Eqs. (16), (17), (18), (19) an (40).

Further experiments are needed to determine the value of m, which can be achieved by rearranging Eq. (47) to fit the data on S and ξ.

It should be noted that the variation of U in Eq. (46) is very large depending on the types of geological materials which determines their hydraulic conductivity, K. For example, de Marsily (1986, p. 78) summarised the values of K for two major types of geological materials, and for sandstone, $K = 1\text{x}10^{-4} \sim 1\text{x}10^{-10}(m/s)$. We use the upper limit of $K = 1\text{x}10^{-4}\, m/s$ to graph Eq. (46) in Figure 5. For Figure 5, a unit hydraulic gradient (Gardner et al., 1970) is assumed to simplify the computation.

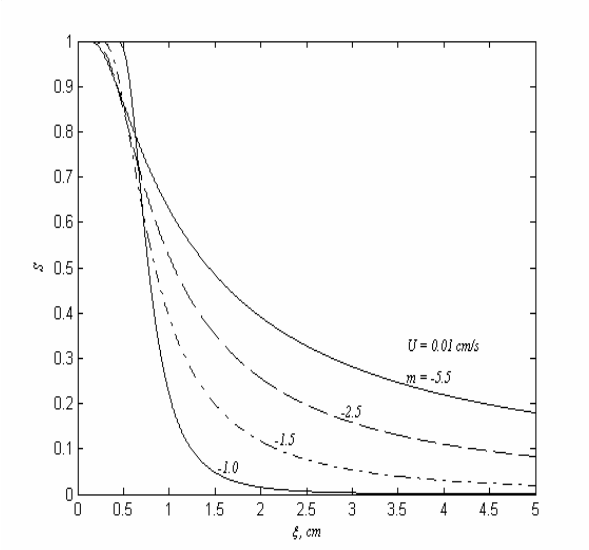

Figure 5. Distribution of saturation across a moving wetting front

Figure 5 shows that at the moving wetting front, the media is saturated while in the rest of the region the saturation follows a nonlinear decrease.

CONCLUSION

In the preceding presentation, we analysed radial two-phase flow of liquid and gas in unsaturated porous media. The following points summarise the presentation:

The equation of radial two-phase flow of liquid and gas (ETPR) in unsaturated porous media is presented, which is a counterpart of McWhorter's equation in Cartesian coordinates. The group transformation given by Philip (1994) transforms the ETPR to a scale-dependent radial two-phase flow equation.

We presented two solutions to the ETPR corresponding to two different instantaneous inputs: one is an exact solution of ETPR in three dimensions for radial two-phase flow into a spherical domain from an instantaneous point source input, and the other an exact solution of ETPR in two dimensions for radial two-phase flow from a well following an instantaneous line source input.

A solution of the ETPR for a continuous input, either in three or two dimensions, can be found by using the instantaneous source solution as a kernel in the convolution integral which incorporates an arbitrary input.

An equation of the wetting front movement is derived by introducing a moving coordinate in the ETPR. This equation of the two-phase wetting front movement in three dimensions is solved, as an example, to obtain an analytical solution subject to the condition for a saturated moving front.

These analytical solutions will aid in the verification of numerical solutions for the simulation of radial two-phase flow in porous media. In particular, the solution of the wetting front movement offers a very simple means to evaluate the fractal parameter, m, of porous media.

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