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Asymptotic Mode-I Crack-tip Stress Fields for Orthotropic Graded Materials

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Summary

Asymptotic analysis coupled with Westergaard stress function approach is used to develop quasi-static stress fields for a crack oriented along one of the principal axes of inhomogeneous orthotropic medium. In the formulation, four independent engineering constants, E_{11}, E_{22}, G_{12}, ν_{12} are replaced by an effective stiffness $E = \sqrt{E_{11}E_{22}}$, a stiffness ratio $\delta = (E_{11}/E_{22})$, an effective Poisson's ratio $\nu = \sqrt{\nu_{12}\nu_{21}}$, and a shear parameter $k = (E/2G_{12}) - \nu$. It is assumed that the effective stiffness varies exponentially along one of the principal axes of orthotropy. The first two terms in the expansion of stress field are derived to explicitly bring out the influence of nonhomogeneity on the structure of the stress field. Using the derived stress field equations, the isochromatic fringe contours are developed to understand the variation of stress field around the crack tip as a function of both orthotropic stiffness ratio and non-homogeneous coefficient.

Introduction

With growing applications of functionally gradient materials (FGMs) [1] research on fracture behavior of these nonhomogeneous solids has generated considerable interest. A large body of work has been reported on general fracture mechanics problems such as behavior of cracks and nature of stress fields in FGMs by several researchers [2-3]. However, for detailed experimental fracture investigation of these materials using techniques such as photoelasticity, moiré interferometry, speckle interferometry and coherent gradient sensing, asymptotic expansion of crack-tip field equations are necessary. Considering a Taylor series expansion of elastic property variation in nonhomogeneous solids, Eischen [3] showed that the stresses proportional to $r^{-1/2}$ and r^0 are not affected by the material property variation and the effect of nonhomogeneity reflects only in the higher order terms ($r^{1/2}$, r, $r^{3/2}$ etc.). Later, Parameswaran and Shukla [4] developed asymptotic expansion of quasi-static crack-tip stress fields for a crack aligned along the direction of property variation in FGMs. In continuation of their studies, Chalivendra et al., [5] obtained quasi-static stress field equations for cracks inclined to the property gradation.

The above mentioned fracture studies were conducted by considering cracks in isotropic graded materials. As it is reported in the literature [6], the graded materials are rarely isotropic because of the nature of techniques used in fabricating

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them. Thus, it is necessary to consider the anisotropic character when studying the failure behaviors of FGMs. In this paper, quasi-static stress fields for a crack oriented along one of the principal axes of inhomogeneous orthotropic medium are developed using Asymptotic analysis coupled with Westergaard stress function approach. It is assumed that the effective stiffness $E = \sqrt{E_{11}E_{22}}$ varies exponentially along one of the principal axes of orthotropy. The first two terms in the expansion of stress field are derived to explicitly bring out the influence of nonhomogeneity on the structure of the stress field. Using the derived stress field equations, the isochromatic fringe contours are developed to understand the variation of stress field around the crack tip as a function of both orthotropic stiffness ratio and nonhomogeneous coefficient.

Theoretical Formulation

The stress function approach of solving plane crack problems leads to a biharmonic equation for homogeneous materials, which is satisfied, by the real and imaginary part of any analytical complex function [7]. However, in the case of FGMs, due to spatial variation of elastic properties, the governing differential equation contains many lower order differential terms as discussed below in this section, making the solution procedure more involved.

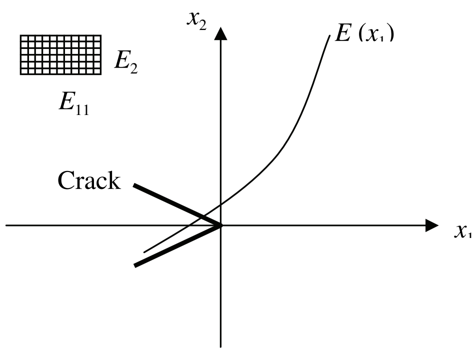


Figure 1: Crack orientation with respect to the direction of effective Young's modulus variation in orthotropic inhomogeneous medium

Consider the plane elasticity problem for an inhomogeneous orthotropic medium containing a crack along $x_2 = 0$, where x_1 and x_2 are the principal axes of orthotropy as shown in Fig.1. Let u_i and σ_{ij} $(i,\ j = 1, 2)$ refer to the displacements and stresses and E_{ii}, G_{ij}, and νij $(i,\ j = 1,\ 2,\ 3)$ refer to the engineering elastic parameters. These four independent material parameters are replaced by the effective Young's modulus E, effective Poisson's ratio ν, the stiffness ratio δ, and the shear parameter κ as defined by Krenk [8].

$$E = \sqrt{E_{11}E_{22}}, \quad \nu = \sqrt{\nu 12 \nu_{21}}, \quad \delta^4 = \frac{E_{11}}{E_{22}} = \frac{\nu 12}{\nu_{21}}, \quad \kappa = \frac{E}{2G_{12}} - \nu \qquad (1)$$

Eq. (1) are valid for plane stress conditions. For plane strain Eq. (1) must be replaced by

$$
\begin{aligned}
E &= \left(\frac{E_{11}E_{22}}{(1-\nu 13\nu 31)(1-\nu 23\nu 32)} \right)^{1/2} \\
\nu &= \left(\frac{(\nu 12+\nu 31\nu 32)(\nu 21+\nu 23\nu 31)}{(1-\nu 13\nu 31)(1-\nu 23\nu 32)} \right)^{1/2}, \\
\delta^4 &= \frac{E_{11}}{E_{22}} \frac{(1-\nu 23\nu 32)}{(1-\nu 13\nu 31)}, \quad \kappa = \frac{E}{2G_{12}} - \nu
\end{aligned}
\tag{2}
$$

In a general inhomogeneous orthtropic materials the parameters E, ν, δ and κ would be functions of x_1 and x_2. However, to simplify the formulation of the problem and to emphasize the spatial variations in the stiffness parameters, the problem is solved under certain restrictive assumptions regarding the distribution of the elastic constants. Previously it was shown that the effect of Poisson's ratio on the stress intensity factors is not very significant [2]. Hence, in the present study Poisson's ratio ν is assumed to be a constant rather than a function of x_1 and x_2. It is also assumed that the material stiffness E_{11}, E_{22}, and G_{12} will vary proportionately. This means that δ and κ may also be considered as constants and the inhomogeneity of the medium may be represented by variation in effective Young's modulus E only. By using now δ as a scaling constant and defining [11]:

$$
\begin{aligned}
&x = \frac{x_1}{\sqrt{\delta}}, \quad y = \sqrt{\delta}x_2, \quad u(x,y) = u_1(x,y)\sqrt{\delta}, \quad \upsilon(x,y) = \frac{u_2(x,y)}{\sqrt{\delta}} \\
&\sigma xx(x,y) = \frac{\sigma 11(x_1,x_2)}{\delta}, \quad \sigma_{yy}(x,y) = \delta\sigma 22(x_1,x_2), \quad \sigma xy(x,y) = \sigma 12(x_1,x_2)
\end{aligned}
\tag{3}
$$

then the strains ε_{xx}, ε_{yy} are ε_{xy} related to ε_{11}, ε_{22} are ε_{12} as below:

$$
\varepsilon xx = \varepsilon 11\delta, \quad \varepsilon yy = \frac{\varepsilon 22}{\delta}, \quad \varepsilon_{xy} = 2\varepsilon 12 \tag{4}
$$

To restrict the consideration to a mode-I crack problem it is assumed that the material parameters are independent of x_2. It is further assumed that $E(x_1,x_2)$ is a monotonically increasing function of x_1 and in the crack region it may approximated by an exponential function as follows:

$$
E(x_1,x_2) = E(x_1) = E_0 e^{\alpha x_1} = E_0 e^{\beta x} \tag{5}
$$

where α is the nonhomogeneity parameter having dimension (Length)$^{-1}$. β is a new non-homogeneous coefficient, $\beta = \alpha\sqrt{\delta}$. Now the Hooke's law is written in terms scaled coordinates (x, y) using effective Young's modulus E, effective Poisson's ratio ν and the shear parameter κ.

$$\begin{aligned} \varepsilon xx &= \frac{1}{E(x,y)}\left(\sigma_{xx} - \nu\sigma yy\right) \\ \varepsilon yy &= \frac{1}{E(x,y)}\left(\sigma_{yy} - \nu\sigma xx\right) \\ \varepsilon xy &= \frac{2(\kappa+\nu)}{E(x,y)}\sigma xy \end{aligned} \tag{6}$$

where $E(x,y) = E_0 e^{\beta x}$.

The in-plane stress components (σ_{ij}, i, j, $\varepsilon\{x,y\}$) can be defined in terms of the Airy's stress function $F(x,y)$, as given in Eq. (7).

$$\sigma xx = \frac{\partial^2 F}{\partial y^2}, \quad \sigma yy = \frac{\partial^2 F}{\partial x^2} \quad \text{and} \quad \sigma xy = -\frac{\partial^2 F}{\partial x \partial y} \tag{7}$$

Expressing the strain components in terms of the stress function through Hooke's law, the compatibility equation for a plane problem can be written as below.

$$\frac{\partial^4 F}{\partial x^4} + 2\kappa\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} + \beta^2\frac{\partial^2 F}{\partial x^2} - \nu\beta^2\frac{\partial^2 F}{\partial y^2} - 2\beta\frac{\partial^3 F}{\partial x^3} - 2\kappa\beta\frac{\partial^3 F}{\partial x \partial y^2} = 0 \tag{8}$$

At this stage, an assumption is made to keep (κ = 1) so that first three terms of the Eq. (8) provides bi-harmonic expression which is identical to the isotropic materials. This assumption (κ = 1) is valid because it was reported by Ozturk and Erdogan [6] that the effect of typical variation of shear parameter (from κ = 1 to κ = 5) on the values of stress intensity factors is insignificant. By setting κ = 1 in the Eq. (8) and after simplification, the compatibility equation take the form as given in Eq. (9). It is noted here that the non-homogeneous coefficient β has orthotropic stiffness ratio δ in the compatibility equations.

$$\nabla^2\left(\nabla^2 F\right) + \beta^2\left(\nabla^2 F\right) - \beta^2(1+\nu)\frac{\partial^2 F}{\partial y^2} - 2\beta\frac{\partial}{\partial x}\left(\nabla^2 F\right) = 0 \tag{9}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

It may be observed from Eq. (9) that the first term is the bi-harmonic term and the additional lower order differentials are due to the non-homogeneity and orthotropy. The solution for Eq. (9) is obtained through an asymptotic analysis [9] coupled with Westergaard's stress function approach [7].

The details of the asymptotic approach are not discussed in this paper and details of procedure can be found in the author's previous paper [5]. The final expressions for the stresses are given below

$$\begin{aligned}\sigma xx = &\sum_{n=0}^{1}\left[\mathrm{Re}\{P_n\} - y\mathrm{Im}\{P_n'\} + 2\mathrm{Re}\{Q_n\}\right] \\ &+ \beta\left[-\mathrm{Re}\{\overline{P_0}\} + 2y\mathrm{Im}\{P_0\} + \frac{y^2}{2}\mathrm{Re}\{P_0'\}\right]\end{aligned} \tag{10}$$

$$\sigma yy = \sum_{n=0}^{1}\left[\mathrm{Re}\{P_n\} + y\mathrm{Im}\{P_n'\}\right] + \beta\left[\frac{-y^2}{2}\mathrm{Re}\{P_0'\}\right] \tag{11}$$

$$\sigma xy = \sum_{n=0}^{1} -\left[y\mathrm{Re}\{P_n'\}\right] + \beta\left[y\mathrm{Re}\{P_0\} - \frac{y^2}{2}\mathrm{Im}\{P_0'\}\right] \tag{12}$$

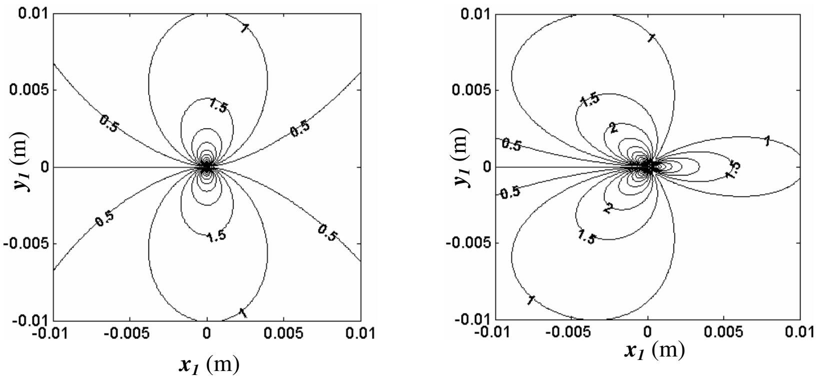
The complex expressions P_n and Q_n, in the (x, y) coordinates cab be again found in author's paper [5]. By using the relation given in equations (3), the above stress field equations (10-12) can be represented in terms of (x_1, x_2) coordinates.

$$\sigma 11(x_1,x_2) = \delta\sigma xx(x,y),\;\; \sigma_{22}(x_1,x_2) = \sigma yy(x,y)/\delta,\;\; \sigma xy(x,y) = \sigma 12(x_1,x_2) \tag{13}$$

The above-developed solutions (10) to (12) are used to study the effect of non-homogeneity and orthotropic stiffness ratio on the structure of stress fields around crack tip. The isochromatic fringe contours associated with constant maximum shear stress near the crack tip are generated for opening mode loading conditions. These contours are related to widely used experimental technique for fracture studies namely photoelasticity [10]. Due to brevity of space, only a typical plot as a function of orthotropic stiffness ratio is shown in figure 2. Figure 2(a) represents isotropic and homogenous case where the fringes are quite familiar without any tilt. However upon addition of orthotropy and non-homogeneity, the fringe size increases and they tilt towards the crack face as shown in figure 2(b). The detailed qualitative analysis of the plots for various combinations of nonhomogeneity and orthotropic stiffness ratios are not discussed in the paper however they will be presented in the conference.

Conclusions

First two terms of quasi-static stress fields for a crack oriented along one of the principal axes of orthotropy are developed using asymptotic analysis. Using these equations, isochromatic contours are generated as a function of orthotropic stiffness ratio and non-homogeneity parameter. Due to the brevity of space, a typical plot is shown in the paper and the several plots representing various combinations of



(a) δ = 1, $\alpha = 0$ (b) δ = 1.5, $\alpha = 5$

Figure 2: Contours of isochromatic contours near the crack-tip for opening mode loading for $K_1 = 1.0$ MPa-m$^{1/2}$

non-homogeneity parameter and stiffness ratio are not discussed in the paper. The detailed analysis of all these plots will be presented at conference. These stress field equations can be used in extracting fracture parameters from the experimental data around the crack-tip in a non-homogeneous orthotropic medium.

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