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Hierarchical Multi-Grid Method for Ultra Large Scale Problem Based on Variational Theorem

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Summary

The authors have proposed Fractal and Hierarchical Multi-Grid Methods for solving ultra large FE problems [1, 2]. In these methods, the domain to be analyzed is subdivided into multi-grid which has fractal or hierarchical structure and the solution is obtained by solving equations for small cells or nodes at each hierarchy successively. In this research, potential capability of a Hierarchical Multi-Grid method is examined through simple example problems.

Introduction

To solve ultra large scale FE problems, iterative solution methods may be superior to the direct solution methods in terms of computing time and required memory. Hierarchical Multi-Grid Method is one of the iterative method in which the approximate solution is obtained through node by node error correction procedure in the hierarchical manner according to the minimum potential energy principle [2]. Its potential capability is demonstrated through two and three dimensional simple problems. Also thermal deformation of cast part after taking out from the mold is analyzed as an example of practical problems.

HMG Based on Minimum Potential Energy Theorem

The proposed HMG is an iterative solution procedure to solve ultra large scale elastic or thermal problems. The model to be analyzed is defined in a cubic grid space which has a hierarchical structure. The displacement or the temperature at a node (or a grid) is computed successively in hierarchical order based on the variational theorem. In case of elastic problem, the displacement u can be computed using the Minimum Potential Energy Theorem which is given by the following functional.

$$\Pi(u) = \frac{1}{2}\int[\sigma]\{\varepsilon\}dv - \int[g]\{u\}dv - \int[\tau]\{u\}ds_\sigma \quad (1)$$

where, u: displacement, σ: stress, g: body force, τ: traction applied as the external load. In HMG, the displacement field u is approximated using the interpolation function with hierarchical structure consists of H levels, i.e.

$$u = \sum_{h=1}^{H} u_h = \sum_{h=1}^{H}[A_h]\{U_h\} \quad (2)$$

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where, $[A_h]$ is the interpolation function and $\{U_h\}$ is the nodal displacement at H-th level. Using Eq. (2), the strain-displacement relation and the stress-strain relation can be described in the following form.

$$\{\varepsilon\} = \sum_{h=1}^{H} [B_h]\{U_h\} \tag{3}$$

$$\{\sigma\} = [D]\{\varepsilon\} = [D]\sum_{h=1}^{H} [B_h]\{U_h\} \tag{4}$$

Substituting Eqs. (3) and (4) into Eq. (1), the functional $\Pi(u)$ can be rewritten in terms of the nodal displacement parameter $\{U_h\}$ with hierarchical structure.

$$\begin{aligned}\Pi(U_1,\ldots,U_h,\ldots,U_H) =& \frac{1}{2}\int\left(\sum_{h=1}^{H}[U_h][B_h]^T\right)[D]\left(\sum_{h=1}^{H}[B_h]\{U_h\}\right)dv \\ &-\int[g]\left(\sum_{h=1}^{H}[A_h]\{U_h\}\right)dv-\int[\tau]\left(\sum_{h=1}^{H}[B_h]\{U_h\}\right)ds_\sigma\end{aligned} \tag{5}$$

Based on the stationality condition of the above functional, iterative solution procedure can be constructed. Let assume, that $\{U_1,\ldots,U_h,\ldots,U_H\}$ is the current approximation and $\{\Delta U_h\}$ is the correction for$\{U_h\}$. The correction vector $\{\Delta U_h\}$ can be obtained through the stationality condition of the following functional.

$$\begin{aligned}&\Pi(U_1,U_h+\Delta U_h,\ldots,U_H) \\ =&\frac{1}{2}\int[\Delta U_h][B_h]^T[D][B_h]\{\Delta U_h\}dv+\int[\Delta U_h][B_h]^T[D]\left(\sum_{h=1}^{H}[B_h]\{U_h\}\right)dv \\ &-\int[\Delta U_h][A_h]^T\{g\}dv-\int[\Delta U_h][B_h]^T\{\tau\}ds_\sigma\end{aligned} \tag{6}$$

Since the displacement parameter $\{U_h\}$ consists that of a set of nodes (from 1 to N_h), the correction is made node by node, i.e.

$$\{U_h+\Delta U_h\} = \left[U_h^1,U_h^2,\ldots,U_h^n+\Delta U_h^n\ldots,U_h^{N_h}\right]^T \tag{7}$$

Thus, the stationality condition can be written as,

$$\begin{aligned}
&\delta\Pi\left(\delta\Delta U_h^n\right)\\
=&\int\left[\delta\Delta U_h^n\right]\left[B_h^n\right]^T[D]\left[B_h^n\right]\left\{\Delta U_h^n\right\}dv+\int\left[\delta\Delta U_h^n\right]\left[B_h^n\right]^T[D]\left(\sum_{h=1}^{H}[B_h]\{U_h\}\right)dv\\
&-\int\left[\delta\Delta U_h^n\right]\left[A_h^n\right]^T\{g\}\,dv-\int\left[\delta\Delta U_h^n\right]\left[A_h^n\right]^T\{\tau\}\,ds_\sigma\\
=&0
\end{aligned}\tag{8}$$

Since Eq. (8) must hold for arbitrary value of $\left\{\delta\Delta U_h^n\right\}$, the following equation is derived.

$$\left\{\Delta U_h^n\right\}=\left[K_h^n\right]^{-1}\left\{f_h^n\right\}\tag{9}$$

where, $\left\{f_h^n\right\}$ is the residual error at the n-th node in the h-th level and,

$$\begin{aligned}
\left\{f_h^n\right\}&=-\int\left[B_h^n\right]^T\{\sigma\}dv+\int\left[A_h^n\right]^T\{g\}dv+\int\left[A_h^n\right]^T\{\tau\}d\\
\left[K_h^n\right]&=\int\left[B_h^n\right]^T[D]\left[B_h^n\right]dv
\end{aligned}\tag{10}$$

The detail of the solution procedure is as follows.

1. As initial values of nodal displacement parameters and stresses, zero is assumed.
2. For the n-th node on the 1-st level, solve Eq. (9) to obtain $\{\Delta U_1^n\}$.
3. Update $\{U_1^n\}$ using $\{\Delta U_1^n\}$.

$$\{U_1^n\}=\{U_1^n+\Delta U_1^n\}\quad(1<n<N_1)\tag{11}$$

4. Using updated $\{U_1\}$, update $\{\sigma\}$ according to Eq. (4).
5. From updated $\{\sigma\}$ and Eq. (9), compute $\{\Delta U_2^n\}$ $(1<n<N_2)$.
6. Repeat 2) through 5) for levels up to h=H.
7. Compute residual error $\left\{e_h^n\right\}$ and its norm E_{rror}.

$$\{e_h^n\}=\{f_h^n\}=-\int\left[B_h^n\right]^T\{\sigma\}dv+\int\left[A_h^n\right]^T\{g\}dv+\int\left[A_h^n\right]^T\{\tau\}ds_\sigma \qquad (1<n<N_h)\tag{12}$$

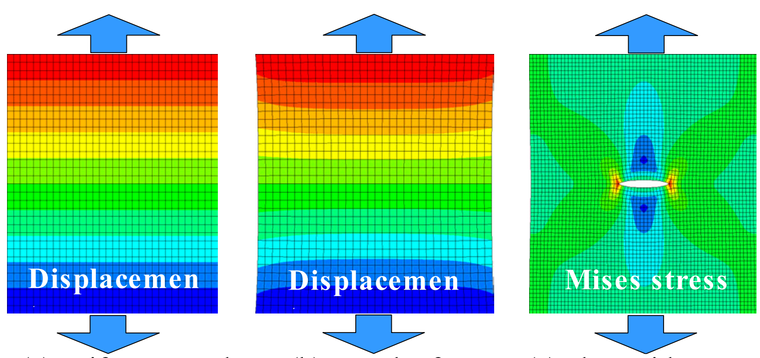
$$E_{rror}=\left(\sum_{h=1}^{H}\sum_{n=1}^{N_h}\{e_h^n\}^T\{e_h^n\}\right)^{1/2}\tag{13}$$

8. Repeat 2) through 7) until the error norm E_{rror} becomes smaller than the tolerance.
9. Substitute the final value of $\{U_h\}$ into Eqs. (2), (3) and (4) to compute displacement u, strain ε and stressσ.

The details of the computational scheme may be different when the problem to be solved is different, such as heterogeneous or nonlinear problems. In case of two and three dimensional elastic problems, Eq. (9) becomes simultaneous equations with two and three unknowns, respectively.

Numerical Examples

Two dimensional elastic problems are taken as examples. The problem (a), (b) and (c) in Fig. 1 are the uniform stretch of a plate, stretch of a clamped plate and that of a plate with center crack. In case (c), the width of plate is 20 mm and the length of crack is 2 mm. Computations are done using Intel Xeon 5160. The computational time is summarized with respect to the degree of freedom (DOF) in Fig. 2. In these computations, the error tolerance is 10^{-5} in terms of the norm defined by Eq. (13) normalized by its initial value. The red line indicates the case in which the computational time is proportional to DOF and it takes one second per 10,000 DOF. As seen from Fig. 2, the computational time is almost proportional to DOF in all three cases and it is roughly 1 second per 10,000 DOF in case of cramped plate. Figure 3 shows the distribution of the stress component in the stretching direction near the crack tip. The red line represents the analytically predicted stress distribution near the crack tip. Regardless of the hierarchical levels, the computed stress distribution agrees well the analytical value.



(a) uniform stretch (b) stretch of clamped plate (c) plate with center crack

Figure 1: Models for two dimensional problems

Second examples are the uniform stretching of a cubic block and the stretch of a clamped block. Though the computational time is larger compared to the two dimensional cases, it is roughly proportional to DOF as shown in Fig.4. In case

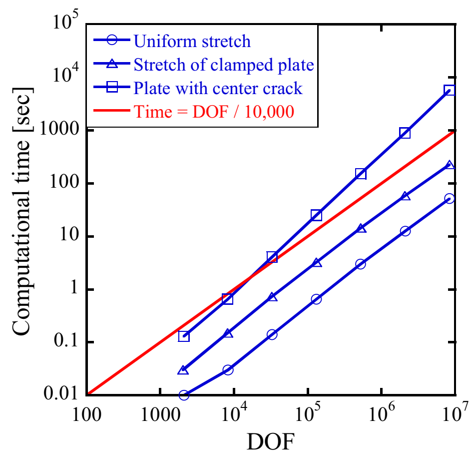


Figure 2: Relation between computational time and DOF (2D problems)

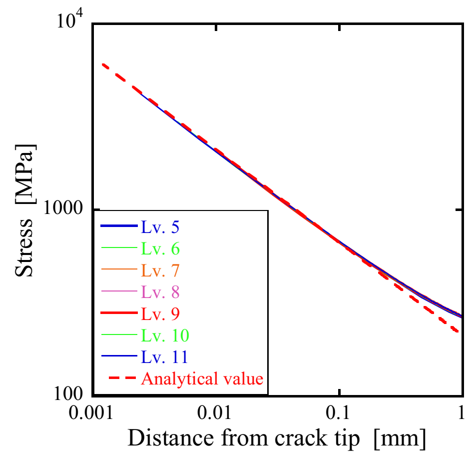


Figure 3: 3 Stress distribution near crack tip

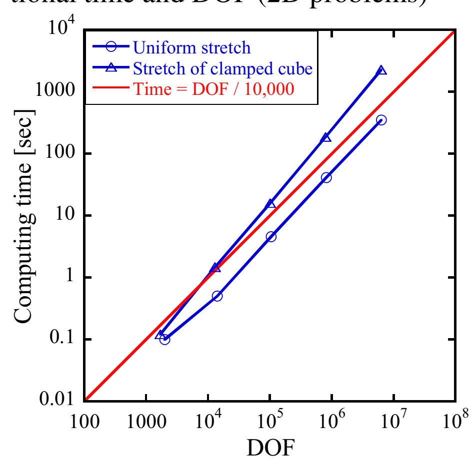


Figure 4: Relation between computational time and DOF (3D problems)

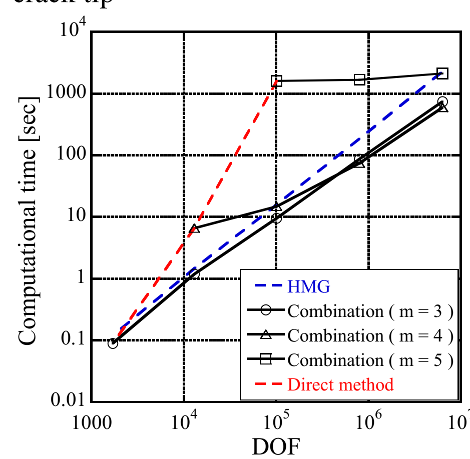


Figure 5: Comparison among direct method, HMG and combined method

of clamped cube, two HMG methods are employed. One is the straight forward HMG and another is the combination of direct solution method and HMG in which the m-levels from the top is computed using direct method and HMG is used for levels lower than m. As seen from Fig. 5, if the level m is appropriately chosen, the computational time can be reduced.

Thermal Deformation of Cast Part

In case of casting, it is necessary to know the thermal deformation of the cast part after taking out from the mold. When the finite difference type mold flow simulation program is employed, it is convenient to use pixel type FEM since the physical values at grid points can be directly transferred to the nodes. The proposed HMG is applied to compute the thermal deformation of a cast part as shown in Fig. 6. The size is 332x168x84 mm. The temperature distribution of the cast part right after taking out from the mold is shown in Fig. 7. Table 1 shows the total degree of freedom and that of elements when HMG with 7, 8 and 9 levels are employed. In case of 9 levels, the total number of nodes which forms the cast part is 1,123,997 while that of the nodes in full cubic space is $513^3 = 1.35 \times 10^8$. The same computations are done using ABAQUS and the computation times are compared with HMG in Table 1. When DOF is 3,371,991, computation was not accepted by ABAQUS. As seen from the table, the computational time by HMG increases roughly linearly with DOF. Figure 8 shows the distribution of the Mises stress computed by HMG and ABAQUS using the model with 8 levels.

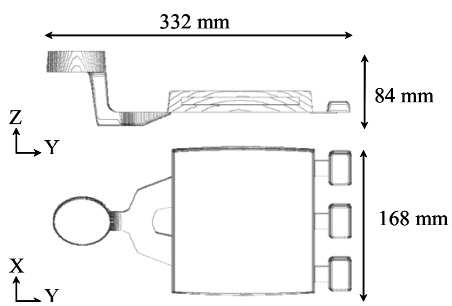


Figure 6: Model of cast part

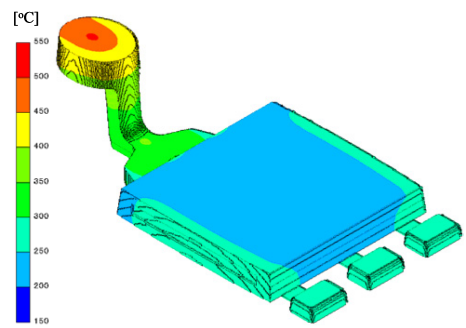


Figure 7: Temperature distribution

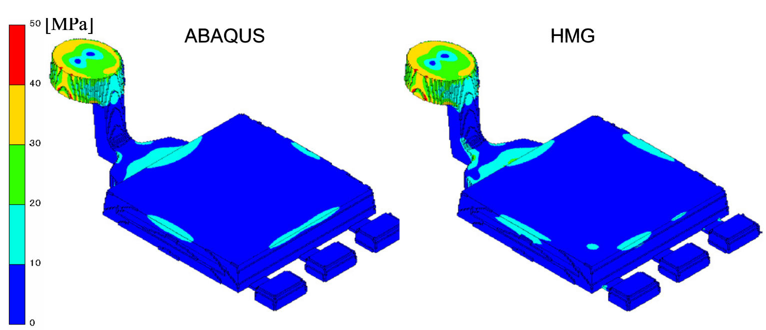


Figure 8: Distribution of Mises stress (DOF=548,247)

Conclusions

Potential capability of the Hierarchical Multi-Grid Method is examined using simple two and three dimensional elastic problems. It is shown that the solution

Table 1: Models and computational time

Level of HMG	Total number of DOF (ratio)	Total number of elements (ratio)	Computational time (sec)	
			HMG (ratio)	ABAQUS (ratio)
7	97,293 (1.0)	22,816 (1.0)	30 (1.0)	166 (1.0)
8	548,247 (5.6)	145,975 (6.4)	217 (7.7)	1567 (9.4)
9	3,371,991 (35)	983,387 (43)	1338 (45)	-

time for simple case is about one second per 10,000 DOF and it is about three seconds per 10,000 DOF in case with complex geometry as the cast part.

References

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