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Multilevel Incomplete Factorizations for Non-Linear FE problems in Geomechanics

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Summary

The geomechanical simulation of a faulted producing reservoir is a strongly non-linear problem due to the non-elastic behaviour of the depleted rocks and especially to the contact constraint prescribed on the fault surfaces. With penalty interface elements for the fault modeling the stiffness matrix may be severely ill-conditioned and the solution by the Preconditioned Conjugate Gradient requires the development and the implementation of ad hoc preconditioners. The present communication investigates the performance of a multilevel incomplete factorization aimed at enhancing convergence and reducing the computational cost of a non-linear geomechanical simulation.

Introduction

The structural behaviour of a producing faulted reservoir can be simulated with the combined use of Finite Elements (FE) and Interface Elements (IE) with a penalty formulation [1]. The resulting numerical problem is strongly non-linear. The reason is twofold. First, while the porous medium outside the reservoir and the connected aquifer can be satisfactorily described by a linear elastic constitutive model, the depleted volume typically exhibits large effective stress variations, hence significant changes of the related stiffness. Second, the opening or slip of fault surfaces is simulated by a non-linear rigid-plastic constitutive model. The final solution is obtained through a linearization procedure.

From a numerical viewpoint, the matrix of the linearized system is symmetric and positive definite (SPD) and may suffer from the ill-conditioning arising from the stiffness contrast between the elements lying inside and outside the depleted porous medium and the presence of large penalty coefficients used to enforce the contact constraints. The efficient solution of these systems by the Preconditioned Conjugate Gradient (PCG) requires special ad hoc preconditioners to accelerate, or even to allow for, convergence. Recently, the Mixed Constraint Preconditioner (MCP) [2] has proved a robust and efficient alternative to more traditional preconditoners. MCP exploits the block structure of the global SPD matrix:

$$\mathscr{A} = \begin{bmatrix} K & B \\ B^T & C \end{bmatrix} \tag{1}$$

where K is the stiffness block of the unknowns linked only to FE and C is the penalty block linked to IE. Unfortunately, when a non-linear constitutive model

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is used for FE the structural block K is not completely invariant and this can deteriorate the MCP performance in a whole simulation. The proposed multilevel incomplete factorization is intended to overcome the drawbacks of the native MCP while perserving its good spectral properties.

The double non-linearity of the above geomechanical problem is addressed by an explicit-explicit approach [1] with the total load subdivided into n steps and applied progressively. At the beginning of each step the jacobian is computed and the linearized system is solved providing the displacement field to be used in the next step. However, to solve accurately the IE non-linearity other substeps are needed at each loading step. Thus, each loading step is further subdivided into m substeps and two nested loops are set up: the part of the stiffness matrix depending on the depleted plastic elements is updated only at each iteration of the outer loop, while the part depending on the IE is updated at every inner iteration. If the unknowns linked to the linear FE are numbered first, those linked to the non-linear FE are numbered second and those linked to the IE are numbered last, the system matrix takes on the following 9 block form:

$$\mathscr{A} = \begin{bmatrix} K_1 & B_{11} & B_{12} \\ B_{11}^T & K_2 & B_2 \\ B_{12}^T & B_2^T & C \end{bmatrix} \tag{2}$$

Note that with this ordering the matrices K_1 and $B_1 = [B_{11}, B_{12}]$ do not change during the whole simulation, K_2 and B_2 change every outer iteration, while C changes every inner iteration.

Multilevel Incomplete Factorization (MIF)

The partial incomplete factorization of a symmetric matrix $\mathscr{A}$ can be defined as:

$$\mathscr{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \simeq \mathscr{M} = \begin{bmatrix} L_1 & 0 \\ H_1 & I \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & S_1 \end{bmatrix} \begin{bmatrix} L_1^T & H_1^T \\ 0 & I \end{bmatrix} = \mathscr{L}_1 \mathscr{D}_1 \mathscr{L}_1^T \tag{3}$$

where L_1 is a lower triangular matrix, H_1 is a rectangular matrix, D_1 is a diagonal matrix and S_1 is in general a full square matrix called Schur complement. Note that $L_1 D_1 L_1^T$ is the incomplete root-free factorization of the A_{11} block, $H_1 = A_{12}^T L_1^{-T} D_1^{-1}$ and $S_1 = A_{22} - H_1 D_1 H_1^T$. The application of $\mathscr{M}^{-1}$ as a preconditioner in a PCG iteration requires the solution to:

$$\mathscr{M}\mathbf{v} = \mathbf{r} \tag{4}$$

that can be carried out with a forward substitution $\mathscr{L}_1 \mathbf{z} = \mathbf{r}$, the solution of a block diagonal system $\mathscr{D}_1 \mathbf{y} = \mathbf{z}$ and a backward substitution $\mathscr{L}_1^T \mathbf{v} = \mathbf{y}$. In the second stage of the procedure the solution of the block diagonal system can be found as follows:

$$\begin{bmatrix} D_1 & 0 \\ 0 & S_1 \end{bmatrix} \left\{ \begin{array}{c} \mathbf{y}_1 \\ \mathbf{y}_2 \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{z}_1 \\ \mathbf{z}_2 \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \mathbf{y}_1 = D_1^{-1}\mathbf{z}_1 \\ \mathbf{y}_2 = S_1^{-1}\mathbf{z}_2 \end{array} \right. \tag{5}$$

The computation of $\mathbf{y}_1$ is straightforward because D_1 is diagonal, while a generally full system with matrix S_1 has to be solved for $\mathbf{y}_2$. This can be done approximately by performing a partial incomplete factorization of S_1, too.

The basic idea of some multilevel preconditioners, e.g. [3] [4] [5] and the present one, is to use recursively a partial incomplete factorization of the Schur complement of each level:

$$S_i \simeq \begin{bmatrix} L_{i+1} & 0 \\ H_{i+1} & I \end{bmatrix} \begin{bmatrix} D_{i+1} & 0 \\ 0 & S_{i+1} \end{bmatrix} \begin{bmatrix} L_{i+1}^T & H_{i+1}^T \\ 0 & I \end{bmatrix} \tag{6}$$

with $L_{i+1}D_{i+1}L_{i+1}^T \simeq S_{i_{11}}$, $H_{i+1} \simeq S_{i_{12}}L_{i+1}^{-T}D_{i+1}^{-1}$ and $S_{i+1} \simeq S_{i_{22}} - H_{i+1}D_{i+1}H_{i+1}^T$, starting with $S_0 = \mathscr{A}$. A dropping strategy similar to that of [6] and [7] is defined to reduce the memory occupation and increase the performance. The fill-in degree is controlled at each level by two parameters:

ρ_i, i.e. the maximum allowable number of nonzeroes for each row of L_{i+1} in excess to those of the 11 block of S_i;

$\tilde{\rho}_i$, i.e. the maximum allowable number of nonzeroes for each row of S_{i+1} in excess to those of the 22 block of S_i.

Such a strategy has the advantage of preserving the nice spectral properties of MCP [2]. Moreover, in conjunction with the explicit-explicit approach described in [1], it allows for a significant saving of CPU time also in the preconditioner computation. As outlined previously, the geomechanical problem of a faulted producing reservoir can be recasted in a three-level block structure. The first level is relative to linear elastic elements, hence the 11 block does not change during the simulation with L_1, D_1 and H_1 computed only once in a pre-processing stage. Furthermore the calculation of the second level block matrices L_2, D_2 and H_2 is performed at each outer loop, so only S_2 and L_3 have to be computed at each inner iteration. This provides an effective, low-memory demanding and relatively cheap preconditioner, as it will be shown in the next section.

Numerical Results

A large size geomechanical problem addressing the deformation of a 3D real faulted gas reservoir is considered. The model discretizes a porous volume with an areal extent of 35x50 km and a depth of 10 km, with fixed bottom and lateral boundaries and a traction-free top surface. Sixteen regional faults have been considered and modelled with the penalty IE developed in [1]. The Young modulus of

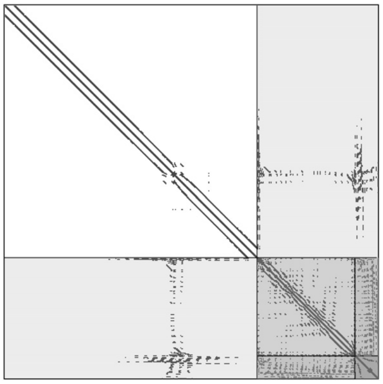

Figure 1: Level structure of the system matrix.

the porous medium varies according to the constitutive law in [8] with the penalty parameters set between 5 and 6 orders of magnitude larger. The number of unknowns belonging to each level is summarized in table 1, while the level structure of the global matrix $\mathscr{A}$ is shown in figure 1.

Table 1: Number of unknowns of each level.

	Level 1	Level 2	Level 3	Total
# of unknowns	435,207	163,581	20,014	618,802

As a benchmark of the new preconditioner (MIF) an incomplete Cholesky factorization (ILLT) of $\mathscr{A}$ has been chosen. Its fill-in degree is controlled by a single parameter ρ_{ILLT} with the same meaning as ρ_i. The performances have been compared in terms of CPU time and memory occupation. All the simulations have been run on a scalar computer equipped with an Intel Core2 Duo processor at 2.13 GHz, 2GB of core memory and 2 MB of secondary cache. The convergence of PCG is achieved when the real relative residual is smaller than 10^{-6}. The memory occupation is evaluated by the preconditioner density μ, i.e. the ratio between the memory needed to store the preconditioner and the system matrix $\mathscr{A}$.

The best time performance of ILLT in the solution of a single system has been obtained setting $\rho_{ILLT} = 70$. Table 2 shows μ_{ILLT}, the number of iterations and the CPU time for the preconditioner calculation and the PCG iterations.

The best time performance of MIF in the solution of a single system has been obtained setting $\rho_1 = 30$, $\tilde{\rho}_1 = 40$, $\rho_2 = 40$, $\tilde{\rho}_2 = 10$ and $\rho_3 = 120$ (note that $\tilde{\rho}_3$ is unneccessary because the level 3 Schur complement is not computed). Table 2 reports the number of iterations, the CPU time to compute each level and the time for the PCG iterations. The memory occupation is evaluated by two parameters $\mu_{MIF}^{(1)}$ and $\mu_{MIF}^{(2)}$, denoting the memory to store the preconditioner and the total amount needed for its computation, respectively. Both indices are helpful because, though the main program requires at least $\mu^{(2)}_{MIF}$, in a real field application part of the memory equal to $\mu^{(2)}_{MIF} - \mu^{(1)}_{MIF}$ may be reused later for other variables. In the solution of a single system MIF outperforms ILLT by a factor 2 requiring slightly less memory. The memory saving can be further increased setting $\rho_1 = 10$, $\tilde{\rho}_1 = 10$, $\rho_2 = 30$, $\tilde{\rho}_2 = 10$ and $\rho_3 = 100$, giving $\mu^{(1)}_{MIF} = 1.89$ and $\mu^{(2)}_{MIF} = 2.35$, though with a slight loss of performance.

Table 2: Comparison of ILLT and MIF memory requirement and best time performance.

	ILLT		MIF
		μ^1_{MIF}	2.68
μ_{ILLT}	3.92	μ^2_{MIF}	3.29
# iter.	75	# iter.	53
		Lev. 1 [s]	23.17
		Lev. 2 [s]	11.29
Fact. [s]	108.69	Lev. 3 [s]	17.04
PCG [s]	36.43	PCG [s]	21.26
Total [s]	145.12	Total [s]	72.76

Note that the CPU time for level 1 and level 2 computations can be actually spread among several non-linear iterations. Compare now the performance provided in a whole transient simulation. The production of the considered faulted gas reservoir has been simulated with 150 outer steps, each one subdivided into 5 substeps for the IE non-linearity. Table 3 shows the total simulation time along with the time spent for calculating the preconditioners, the time for the PCG iterations and the overhead, i.e. the total time spent for the system assembly and the elements update. The overhead is negligibly small and the PCG iteration time is almost the same for the two approaches. The main advantage of MIF is that it is a much cheaper preconditioner with the same iteration count as ILLT. In this real field example, MIF compares very favourably with ILLT showing a total speed-up of 3.5.

Table 3: ILLT and MIF time performance in a real field transient simulation.

	ILLT	MIF
Prec. Calc.	26h 57m 33s	4h 44m 10s
PCG Iter.	3h 45m 42s	3h 41m 06s
Overhead	22m 45s	21m 13s
Total	31h 06m 0s	8h 46m 29s

Conclusions

The non-linear problem arising from a FE-IE discretization of a faulted producing gas reservoir shows typically ill-conditioned stiffness matrices whose efficient solution by PCG requires the use of special ad hoc preconditioners. Block constraint preconditioners have shown to provide a good performance in problems where linear FE are combined with non-linear IE, however they cannot be as efficient when dealing with non-linear FE due to the relatively high pre-processing cost. In the present communication a three-level incomplete factorization has been developed with almost the same spectral properties as constraint preconditioners but cheaper to compute. In the solution of any single system MIF provides a better time performance than the general algebraic preconditioner ILLT. The speed-up is even larger in a complete transient simulation because its setup is made up for in several non-linear iterations. Hence MIF appears to be a quite promising and cost-effective tool for the iterative solution to non-linear multilevel problems.

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