Toward Structural Integrity Analyses based on a Temporal Multisclae Scheme

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Summary

In this paper, a method for performing fatigue crack propagation analyses based on a temporal multiscale scheme is presented. Although plastic deformation around the region of crack tip occurs when a cracked structure is subject to a fatigue load, many fatigue crack propagation analyses have assumed linear elasticity. The total number of load cycles may be in the range of $10^3 \sim 10^9$. It is computationally too intensive to follow the nonlinear deformation histories during every load cycle. In proposed temporal-multiscale approach, detailed elastic-plastic analysis performed to follow the deformation histories during a load cycle. It is called "micro-temporal scale" analyses. Analysis based on "macro-temporal scale" is carried out and a hundreds of load cycles are skipped. Then, the "micro-temporal scale" analysis is performed again. The "macro-temporal scale analysis" follows.

Multi-scale analyses have attracted many researchers and engineers in the field of computational solid mechanics. Numerical homogenization technique (see [1, 2, 3] for example) based on two-scale representation has been applied to many problems with assuming that the micro-structure of solid be spatially periodic. The concept of temporal multi-scale analysis was recently proposed by Fish and Oskay [4] and Yu and Fish [5].

In present research, our goal is to perform fatigue crack propagation analyses. A simple temporal-multiscale scheme that enables us to skip many load cycles has been developed. It is implemented in a finite element program and some preliminary example problems were solved. In this paper, the outlines of the temporal multi-scale approach to the fatigue crack propagation analysis are presented.

Temporal Multi-Scale Formulation

A physical quantity under a fatigue loading ϕ is described as a function of spatial coordinates $\mathbf{x}$ and time t and is written to be:

$$\phi = \phi^{\zeta}(\mathbf{x}, t) \tag{1}$$

Here the superscript ζ indicates that the quantity ϕ is periodic or almost periodic in temporal scale. If $\phi = \phi^{\zeta}(\mathbf{x}, t)$ were periodic in temporal scale, it satisfies the following relationship.

$$\phi = \phi^{\zeta}(\mathbf{x}, t) = \phi(\mathbf{x}, t, \tau) \text{ and } \phi(\mathbf{x}, t, \tau + \kappa) = \phi(\mathbf{x}, t, \tau) \quad (t = \zeta\tau + c) \tag{2}$$

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where τ and t are the fast and slow temporal scales. κ represents the duration of one load cycle in the fast temporal scale τ. ζ represents the duration of one load cycle in the slow temporal scale t. c is a constant. Functions in the form of equation (2) is "periodic" as depicted in Fig. 1 (a). However, in many cases, although the function has a periodic nature it often changes gradually, as shown in Fig. 1 (b). Such a function is called an almost periodic function in temporal scale, and is written to be:

$$\phi = \phi^{\zeta}(\mathbf{x},t) = \phi(\mathbf{x},t,\tau) \text{ and } \phi(\mathbf{x},t,\tau+\kappa) = \phi(\mathbf{x},t,\tau) + \bar{\phi}(x,t)\,\zeta\kappa \tag{3}$$

$\bar{\phi}(\mathbf{x},t)$ represents slow change in physical quantity ϕ and $\zeta\kappa$ is the duration of one load cycle in the slow time scale t.

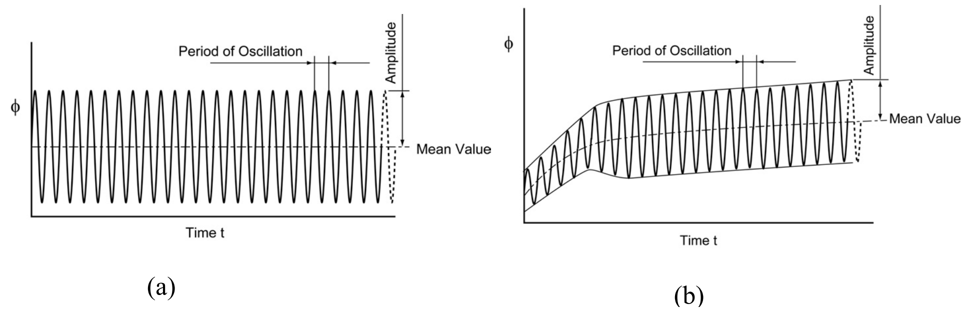


Figure 1: The behaviors of (a) periodic function and (b) almost periodic function

Since the average value and the amplitude of applied cyclic load may gradually change, the physical quantity is considered to be an almost periodic function. Thus, we write the physical quantity in the following fashion.

$$\phi^{\zeta}(\mathbf{x},t) = \phi(\mathbf{x},t,\tau) = M(\phi(\mathbf{x},t,\tau))(\mathbf{x},t) + \tilde{\phi}(\mathbf{x},t,\tau) \tag{4}$$

where $M(\phi(\mathbf{x},t,\tau))(\mathbf{x},t)$ is the value that only depends on the macro-temporal scale t, which corresponds to the mean value of oscillatory behavior of $\phi^{\zeta}(\mathbf{x},t)$. $\tilde{\phi}(\mathbf{x},t,\tau)$ is introduced to represent the oscillating part of the quantity $\phi^{\zeta}(\mathbf{x},t)$. $M(\phi(\mathbf{x},t,\tau))(\mathbf{x},t)$ is defined by the "almost periodic temporal homogenization" (APTH) operator (see [4]), as:

$$\frac{\mathrm{d}M(\phi(\mathbf{x},t,\tau))(\mathbf{x},t)}{\mathrm{d}t} = \frac{1}{\kappa}\int_{\kappa} \dot{\phi}(\mathbf{x},t,\tau)\,\mathrm{d}\tau \tag{5}$$

Here κ denotes a period of oscillation in the micro-temporal scale τ.

Governing Equations and BVP in the Temporal Multiscale Formulation

In the temporal multiscale formulation, the displacements, strains, plastic strains, stresses, body forces and applied tractions are denoted as follows.

Displacements: $u_i^{\zeta}(\mathbf{x},t) = u_i(\mathbf{x},t,\tau) = M(u_i(\mathbf{x},t,\tau))(\mathbf{x},t) + \tilde{u}_i(\mathbf{x},t,\tau)$ (6)

Strains: $\varepsilon_{ij}^{\zeta}(\mathbf{x},t) = \varepsilon_{ij}(\mathbf{x},t,\tau) = M(\varepsilon_{ij}(\mathbf{x},t,\tau))(\mathbf{x},t) + \tilde{\varepsilon}_{ij}(\mathbf{x},t,\tau)$ (7)

Plastic strains: $\varepsilon_{ij}^{p\zeta}(\mathbf{x},t) = \varepsilon_{ij}^{p}(\mathbf{x},t,\tau) = M\left(\varepsilon_{ij}^{p}(\mathbf{x},t,\tau)\right)(\mathbf{x},t) + \tilde{\varepsilon}_{ij}^{p}(\mathbf{x},t,\tau)$ (8)

Stresses: $\sigma_{ij}^{\zeta}(\mathbf{x},t) = \sigma_{ij}(\mathbf{x},t,\tau) = M(\sigma_{ij}(\mathbf{x},t,\tau))(\mathbf{x},t) + \tilde{\sigma}_{ij}(\mathbf{x},t,\tau)$ (9)

Body forces ($b_i^{\zeta}(\mathbf{x},t)$), applied tractions on $\partial\Omega_P$ ($\bar{P}_i^{\zeta}(\mathbf{x},t)$) and prescribed displacements on $\partial\Omega_u$ ($\bar{u}_i^{\zeta}(\mathbf{x},t)$) are also written in the same manner.

The governing equations for the boundary value problem are written using the mean values such as $M(\sigma_{ij}(\mathbf{x},t,\tau))(\mathbf{x},t)$ and the oscillating part such as $\tilde{\sigma}_{ij}(\mathbf{x},t,\tau)$ separately. Elastic-plastic constitutive equation is usually written in a rate form. The rate form stress-strain relationship in terms of macro- and micro- temporal scales, as:

$$\frac{\partial M(\sigma_{ij}(\mathbf{x},t,\tau))(\mathbf{x},t)}{\partial t} = E_{ijk\ell}\left\{\frac{\partial M(\varepsilon_{k\ell}(\mathbf{x},t,\tau))(\mathbf{x},t)}{\partial t} - \frac{\partial M\left(\varepsilon_{k\ell}^{p}(\mathbf{x},t,\tau)\right)(\mathbf{x},t)}{\partial t}\right\} \tag{10}$$

$$\frac{\partial \tilde{\sigma}_{ij}(\mathbf{x},t,\tau)}{\partial \tau} = E_{ijk\ell}\left\{\frac{\partial \tilde{\varepsilon}_{k\ell}(\mathbf{x},t,\tau)}{\partial \tau} - \frac{\partial \tilde{\varepsilon}_{k\ell}^{p}(\mathbf{x},t,\tau)}{\partial \tau}\right\} \tag{11}$$

where $E_{ijk\ell}$ are the components of fourth order tensor that represent the Hooke's law. Evolution equation for the plastic strains can be written in the following form.

$$\frac{\partial \tilde{\varepsilon}_{ij}^{p}}{\partial \tau} = F_{ijk\ell}\left(M\left(\sigma'_{mn}\right), \tilde{\sigma}'_{mn}, \lambda_1, \lambda_2, \cdots\right)\frac{\partial \tilde{\varepsilon}'_{k\ell}}{\partial \tau} \tag{12}$$

$F_{ijk\ell}\left(M(\sigma'_{mn}), \tilde{\sigma}'_{mn}, \lambda_1, \lambda_2, \cdots\right)$ are the functions of deviatoric part of stresses $M(\sigma'_{mn})$, and $\tilde{\sigma}'_{mn}$, and of strain history parameters $\lambda_1, \lambda_2, \cdots$. The functional shapes of $F_{ijk\ell}\left(M(\sigma'_{mn}), \tilde{\sigma}'_{mn}, \lambda_1, \lambda_2, \cdots\right)$ depend on the type of constitutive law which is adopted.

The most important outcome is in the evolution equation for the plastic strains in the macro-temporal scale t. It is derived through averaging processes by using the properties of the APTH operator that is shown in equation (5).

$$\frac{\partial M\left(\varepsilon_{ij}^{p}\right)}{\partial t} = N\left[\tilde{\varepsilon}_{ij}^{p}(\mathbf{x},t,\kappa) - \tilde{\varepsilon}_{ij}^{p}(\mathbf{x},t,0)\right] \tag{13}$$

where N is the number of cycles per unit time under the macro-temporal scale t.

Table 1: BVPs under the micro- and macro-temporal scales

	Micro-temporal scale	Macro-temporal scale
Equilibrium Eq.	$\dfrac{\partial \tilde{\sigma}_{ij}(\boldsymbol{x},t,\tau)}{\partial x_i} + \tilde{b}_j(\boldsymbol{x},t,\tau) = 0$	$\dfrac{\partial M(\sigma_{ij}(\boldsymbol{x},t,\tau))(\boldsymbol{x},t)}{\partial x_i} + M(b_j(\boldsymbol{x},t,\tau))(\boldsymbol{x},t) = 0$
Constitutive law	$\dfrac{\partial \tilde{\sigma}_{ij}(\boldsymbol{x},t,\tau)}{\partial \tau} = E_{ijk\ell}\left\{\dfrac{\partial \tilde{\varepsilon}_{k\ell}(\boldsymbol{x},t,\tau)}{\partial \tau} - \dfrac{\partial \tilde{\varepsilon}^p_{k\ell}(\boldsymbol{x},t,\tau)}{\partial \tau}\right\}$ $\dfrac{\partial \tilde{\varepsilon}^p_{ij}}{\partial \tau} = F_{ijk\ell}(M(\sigma'_{mn}), \tilde{\sigma}'_{mn}, \lambda_1, \lambda_2, \cdots)\dfrac{\partial \tilde{\varepsilon}'_{k\ell}}{\partial \tau}$	$\dfrac{\partial M(\sigma_{ij}(\boldsymbol{x},t,\tau))(\boldsymbol{x},t)}{\partial t} = E_{ijk\ell}\left\{\dfrac{\partial M(\varepsilon_{k\ell}(\boldsymbol{x},t,\tau))(\boldsymbol{x},t)}{\partial t} - \dfrac{\partial M(\varepsilon^p_{k\ell}(\boldsymbol{x},t,\tau))(\boldsymbol{x},t)}{\partial t}\right\}$ $\dfrac{\partial M(\varepsilon^p_{ij})}{\partial t} = N\left[\tilde{\varepsilon}^p_{ij}(\boldsymbol{x},t,\kappa) - \tilde{\varepsilon}^p_{ij}(\boldsymbol{x},t,0)\right]$
Traction BC	$\tilde{\bar{P}}_i(\boldsymbol{x},t,\tau)$	$M(\bar{P}_i(\boldsymbol{x},t,\tau))(\boldsymbol{x},t)$
Disp. BC	$\tilde{\bar{u}}_i(\boldsymbol{x},t,\tau)$	$M(\bar{u}_i(\boldsymbol{x},t,\tau))(\boldsymbol{x},t)$
Body force	$\tilde{b}_i(\boldsymbol{x},t,\tau)$	$M(b_i(\boldsymbol{x},t,\tau))(\boldsymbol{x},t)$

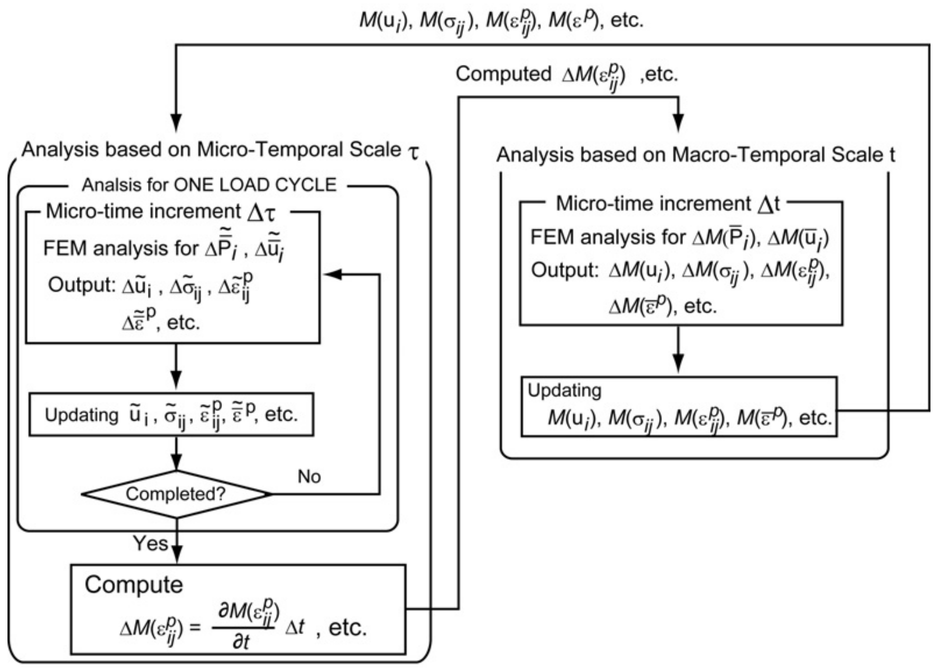


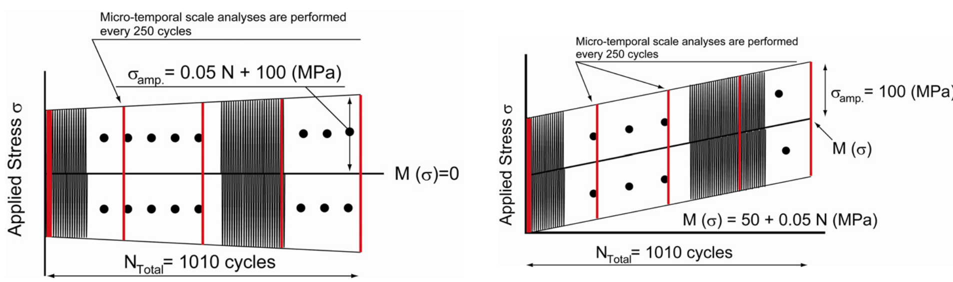
Figure 2: The outline of temporal multiscale algorithm (two-step algorithm)

Solution Algorithm of Temporal Multiscale Analysis

As seen in previous section, the rate from constitutive equations and the evolution equations for the plastic strains are established for macro- and micro-temporal scales, as summarized in Table 1 along with the boundary conditions and body forces. Thus, the boundary value problem is solved by the finite element method. A micro-/macro-temporal scale coupling algorithm can be constructed as shown in Figure 2. A cyclic load cycle is solved and the histories of nonlinear deformation are stored. Then, the rates of plastic strains with respect to the macro-temporal scale are calculated by equation (13). The plastic strain increments are treated as a initial strain when the boundary value problem with respect to the macro-temporal scale is solved.

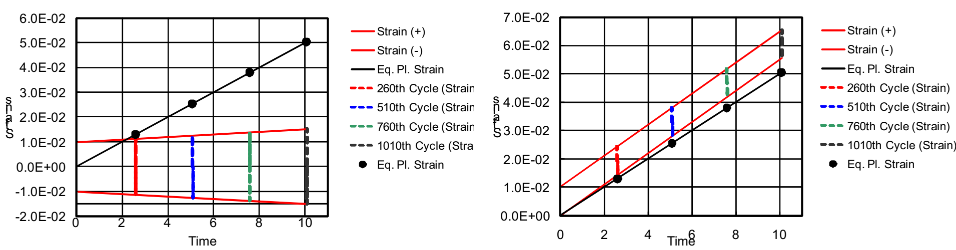
Demonstration Problems

In this section, the results of some simple demonstration problems are presented. A bar is subject to almost periodic fatigue loadings. One has gradually increasing amplitude of oscillating applied stress and the mean value is zero. The other case has a constant amplitude of oscillating applied stress and the mean value gradually increases. They are shown as shown in Figure 3. The J2-Flow theory is adopted for the elastic-plastic analysis. The Young'S modulus and yield stress are set to be 10000 MPa and 100 MPa. The linear hardening low is assumed and the hardening modulus is set to be 100 MPa.



(a) (b)

Figure 3: The almost periodic fatigue ladings [(a) gradually increasing amplitude and (b) gradually increasing mean value]



(a) (b)

Figure 4: Computed strain and plastic strain for the cases of (a) gradually increasing amplitude and (b) gradually increasing mean value of almost periodic fatigue loads. [Lines for strain (+) and strain (-) indicate the highest and the lowest strains in a load cycle that were computed an ordinary time marching scheme with following every load cycle. Line of Eq. Pl. strain shows the equivalent plastic strain that is also computed by an ordinary scheme. Dotted lines and filled circles are the ones computed by the temporal multiscale scheme.]

Ordinary incremental analyses which follow entire loading histories were performed to obtain reference solutions. The results of present temporal-multiscale analyses compared favorably with the reference solutions.

Concluding Remarks

In this paper, a simple methodology to perform temporal-multiscale analysis under fatigue loadings is presented. The results of simple one dimensional problems are presented. The authors are now conducting trials to develop a successful strategy to perform the temporal-multiscale analyses along with the nodal release technique that is often used in crack propagation analyses.

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References

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