High-Order Absorbing Boundaries for Time-Dependent Waves

D. Givoli[1], V. J. van Joolen[2], B. Neta[3]

Summary

Recently developed non-reflecting boundary conditions are applied for exterior time-dependent wave problems in unbounded domains. The linear time-dependent wave equation, with or without a dispersive term, is considered in an infinite domain. The infinite domain is truncated via an artificial boundary $\mathcal{B}$, and a high-order Non-Reflecting Boundary Condition (NRBC) is imposed on $\mathcal{B}$. Then the problem is solved numerically in the finite domain bounded by $\mathcal{B}$. The new boundary scheme is based on a reformulation of the sequence of NRBCs proposed by Higdon. In contrast to previous papers using similar formulations, here the method is applied to a fully exterior two-dimensional problem, with a rectangular boundary. Numerical examples in infinite domains are used to demonstrate the performance and advantages of the new method.

Introduction

Methods for the numerical solution of wave problems in unbounded domains have been developed since the 70's [1]. They have been considered in various fields of application involving wave propagation, such as acoustics, electromagnetics, meteorology, oceanography and geophysics of the solid earth. The four main types of methods that have emerged are: boundary integral methods, infinite element methods, absorbing layer methods and non-reflecting boundary condition (NRBC) methods. The present paper concentrates on the latter.

In the method of NRBCs, the infinite domain is truncated via an artificial boundary $\mathcal{B}$, thus dividing the original domain into a finite computational domain Ω and a residual infinite domain D. A special boundary condition is imposed on $\mathcal{B}$, in order to complete the statement of the problem in Ω (i.e., make the solution in Ω unique) and, most importantly, to minimize spurious wave reflection that result from $\mathcal{B}$. This boundary condition is called a NRBC, although other names such as absorbing, radiating, open, silent, transmitting, transparent, free-space and pulled-back boundary conditions, are often used too [2]. The problem is then solved numerically in Ω. The setup is illustrated in Fig. 1. In the example shown, $\mathcal{B}$ is the boundary on all four sides of Ω, i.e., $\mathcal{B} = \Gamma_E \cup \Gamma_N \cup \Gamma_W \cup \Gamma_S$.

Naturally, the quality of the numerical solution strongly depends on the properties of the NRBC employed. In the last 25 years or so, much research has been done to develop NRBCs that after discretization lead to a scheme which is stable, accurate, efficient and easy to implement. See [3]–[5] for recent reviews on the subject. Of course, it is difficult

[1]Department of Aerospace Engineering, and Asher Center for Space Research, Technion — Israel Institute of Technology, Haifa 32000, Israel
[2]Department of Mathematics, US Naval Academy, 121 Blake Road, Annapolis, Maryland 21402
[3]Department of Mathematics, Naval Postgraduate School, 1141 Cunningham Road, Monterey, CA 93943, USA

Proceedings of the 2004 International Conference on Computational & Experimental Engineering & Science 26-29 July, 2004, Madeira, Portugal

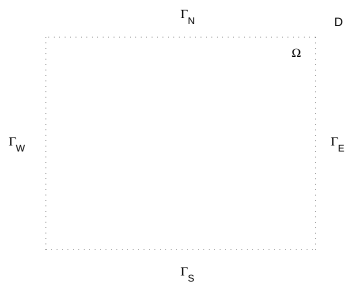

Figure 1: Setup for the NRBC method: an infinite domain problem.

to find a single NRBC which is ideal in all respects and all cases; this is why the quest for better NRBCs and their associated discretization schemes continues.

The late 70's and early 80's produced some low-order local NRBCs that become well-known, e.g., the Engquist-Majda NRBCs [6] and the Bayliss-Turkel NRBCs [7]. More recently, *high-order* local NRBCs have been introduced. Sequences of increasing-order NRBCs have been available before (e.g., the Bayliss-Turkel conditions [7] constitute such a sequence), but they had been regarded as impractical beyond 2^{nd} or 3^{rd} order from the implementation point of view. Only since the mid 90's have practical high-order NRBCs been devised.

Such high-order NRBCs have been proposed by Collino [8], Grote and Keller [9], Sofronov [10], Hagstrom and Hariharan [11], Guddati and Tassoulas [12], Givoli [13], and Givoli *et al.* [14],[15]. In the latter papers, we develop high-order NRBC schemes for both dispersive and non-dispersive linear time-dependent waves. The schemes are based on NRBCs which were originally proposed by Higdon [16]. The original implementation of the Higdon NRBCs was limited to low orders. In [14], Givoli and Neta proposed a new implementation method that allows the use of high-order discretized Higdon NRBCs. However, this method differs from the original Higdon formulation only on the discrete level, not on the continuous level; thus, like the original Higdon scheme, it involves *high normal and temporal derivatives*, of increasing order. This has several clear disadvantages. In addition, the computational effort required by the scheme devised in [14] grows exponentially with the order of the NRBC. In a follow-on paper, Givoli and Neta [15] *reformulated* the Higdon NRBCs on the continuous level in a completely new way. This formulation *does not involve any high derivatives*. This is made possible by introducing special auxiliary variables on $\mathcal{B}$. The new construction allows the easy use of a Higdon-type NRBC of *any* desired order, and can be incorporated in a Finite Element (FE) or a Finite Difference (FD) scheme. In [15] and [17] Givoli et al. used, respectively, FDs and FEs to discretize both the partial differential equation in Ω and the NRBC on $\mathcal{B}$. The computational effort required by the scheme grows only *linearly* with the order.

The development in [14], [15] and [17] was limited to the configuration of a *wave* *guide*, where $\mathcal{B}$ is a single plane boundary (say Γ_E in Fig. 1). In the present paper, we extend the results of Givoli and Neta [14],[15] to the case that all four sides of the rectangular domain Ω require NRBCs. In this case $\mathcal{B}$ has *corners* which may potentially lead to numerical instabilities. In the present paper no special treatment is applied to the corners; indeed a long-time instability is observed for the auxiliary-variable formulation as expected. Special corners conditions may be devised similarly to the treatment in [18]; however, this issue is beyond the scope of this paper.

Statement of the Problem

We consider wave propagation in a two-dimensional infinite domain as described in Fig. . Consider the linear inhomogeneous Klein-Gordon equation,

$$\frac{\partial^2\eta}{\partial t^2} - C_0^2\left(\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2}\right) + f^2\eta = S\,. \tag{1}$$

In (1), η is the unknown wave field, C_0 is the given reference wave speed, f is the given dispersion parameter, and S is a given wave source function. The C_0 and f are allowed to be functions of location in a finite region, outside of which they are constant. The wave source S is a function of location and time, but it is assumed to have a local support.

The initial conditions

$$\eta(x,y,0) = \eta_0 \qquad , \qquad \frac{\partial\eta(x,y,0)}{\partial t} = \eta_0'\,, \tag{2}$$

are given at time $t = 0$ in the entire domain. We assume that the functions η_0 and η_0' have a local support.

We now truncate the infinite domain by introducing an artificial boundary $\mathcal{B} = \Gamma_E \cup \Gamma_N \cup \Gamma_W \cup \Gamma_S$; see Fig. . This boundary divides the original infinite domain into two subdomains: an exterior domain D, and a finite computational domain Ω which is bounded by Γ_W, Γ_N, Γ_S and Γ_E. We choose the location of the four sides such that the entire support of S, η_0, η_0', as well as the region of non-uniformity of C_0 and f, are all contained inside Ω. Thus, on $\mathcal{B}$ and in D, the homogeneous counterpart of (1) holds, i.e.,

$$\frac{\partial^2\eta}{\partial t^2} - C_0^2\left(\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2}\right) + f^2\eta = 0\,, \tag{3}$$

with constant coefficients C_0^2 and f^2, and the medium is initially at rest.

To obtain a well-posed problem in the finite domain Ω we need to impose a boundary condition on $\mathcal{B}$. This must be a NRBC so as to minimize spurious reflection of waves. We use two reformulations of the *Higdon NRBC* [16] that were developed in [14] and [15].

The first one differs from the original Higdon formulation only on the discrete level, and as such involves high order normal and temporal derivatives. The second one only requires second order derivatives. The Higdon NRBC of order J is

$$H_J: \qquad \left[\prod_{j=1}^{J}\left(\frac{\partial}{\partial t}+C_j\frac{\partial}{\partial \nu}\right)\right]\eta = 0 \qquad \text{on} \quad \mathcal{B}\,. \tag{4}$$

Here, $\partial/\partial\nu$ is the normal derivative and the C_j are constant parameters which have to be chosen and which signify phase speeds in the direction normal to the boundary.

The main advantages of the Higdon conditions were listed in Givoli and Neta [15]. Difficulties associated with their original formulation are as follows: (a) The discrete Higdon conditions were developed in the literature up to third order only, because of their *algebraic complexity* which increases rapidly with the order; (b) The original Jth-order Higdon NRBC involves *high normal and temporal derivatives*, up to order J, which pose obvious disadvantages.

The new formulations presented in [14] and [15] overcome all these difficulties. Both reformulations allow the easy use of Higdon NRBCs up to an *arbitrarily high order*. The scheme is coded once and for all for any order; the order of the scheme is simply an input parameter.

Numerical Experiment

We consider an example involving a persistent point source which is turned on at $t = 0$ at the middle of the computational domain. The computational parameters are $\Delta x = \Delta y = .25$, and $\Delta t = .1$. The parameters $C_0 = 1$ and $f = 0$ are used. The Higdon conditions are applied along all four sides of the domain, with $C_j = 1$ for all the j's. The reference domain $\mathcal{D}^*$ is taken here to be large enough that during the computation time $0 \le t \le 6$ the wave front does not reach the extended outer boundaries at all (although it does of course pass the truncated boundary $\mathcal{B}$). We measure the relative global error as a function of time. Fig. 2 shows the maximum relative error during $0 \le t \le 6$ as a function of the Higdon order J, for $1 \le J \le 20$. The error reduces sharply when passing from $J = 1$ (the Sommerfeld-like condition) to $J = 2$, then oscillates slightly when J is further increased, and levels off at about 2.5%. The error cannot be reduced further without also refining the grid and choosing a smaller time-step size. With both Higdon formulations, no instability has been observed in this case. For additional examples where the error is measured for increasing J, see [14] and [15].

Fig. 3 shows the comparison of the computed solution with the reference solution with $J = 20$ at $t = 6$. Very good agreement between the two solutions is observed.

Reference

1. D. Givoli, *Numerical Methods for Problems in Infinite Domains*, Elsevier, Amsterdam, 1992.

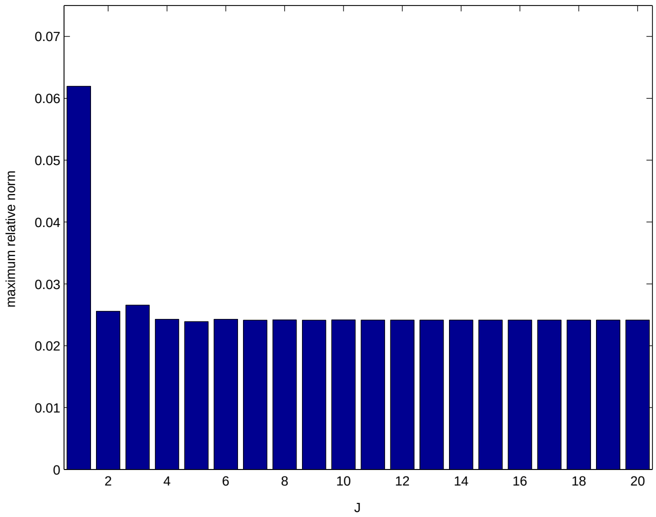

Figure 2: Maximum relative error for $1 \leq J \leq 20$.

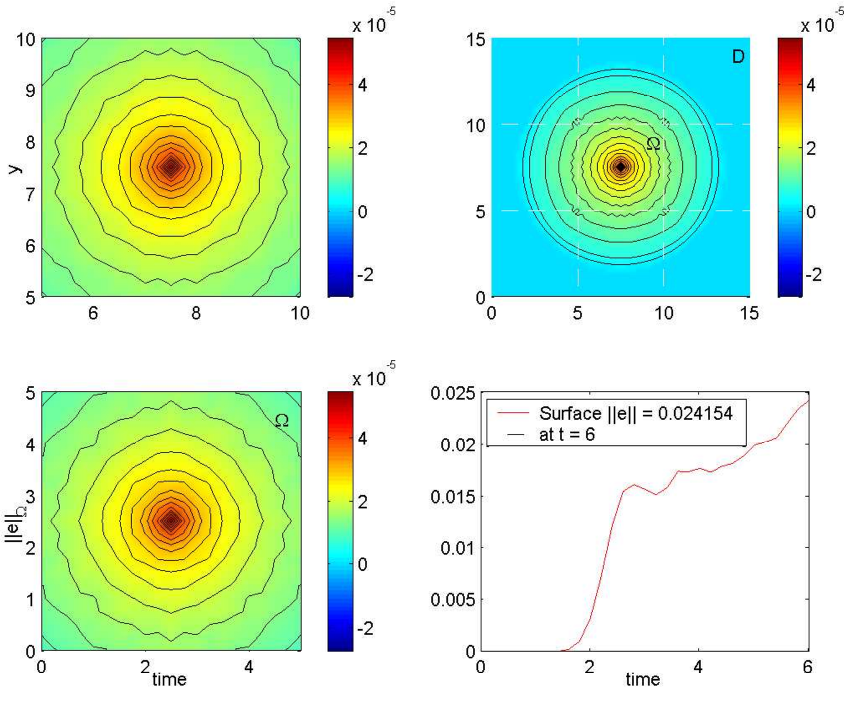

Figure 3: The persistent source problem: solution with $J = 20$ at $t = 6$.

2. D. Givoli, "Non-Reflecting Boundary Conditions: A Review," *J. Comput. Phys.*, **94**, 1–29, 1991.
3. S.V. Tsynkov, "Numerical Solution of Problems on Unbounded Domains, A Review," *Appl. Numer. Math.*, **27**, 465–532, 1998.
4. D. Givoli, "Exact Representations on Artificial Interfaces and Applications in Mechanics," *Appl. Mech. Rev.*, **52**, 333–349, 1999.
5. T. Hagstrom, "Radiation Boundary Conditions for the Numerical Simulation of Waves," *Acta Numerica*, **8**, 47–106, 1999.
6. B. Engquist and A. Majda, "Radiation Boundary Conditions for Acoustic and Elastic Calculations," *Comm. Pure Appl. Math.*, **32**, 313–357, 1979.
7. A. Bayliss and E. Turkel, "Radiation Boundary Conditions for Wave-Like Equations," *Comm. Pure Appl. Math.*, **33**, 707–725, 1980.
8. F. Collino, "High Order Absorbing Boundary Conditions for Wave Propagation Models. Straight Line Boundary and Corner Cases," in *Proc.* 2^{nd} *Int. Conf. on Mathematical & Numerical Aspects of Wave Propagation*, R. Kleinman et al., Eds., SIAM, Delaware, pp. 161-171, 1993.
9. M.J. Grote and J.B. Keller, "Nonreflecting Boundary Conditions for Time Dependent Scattering," *J. Comput. Phys.*, **127**, 52–65, 1996.
10. I.L. Sofronov, "Artificial Boundary Conditions of Absolute Transparency for Two and Three Dimensional External Time Dependent Scattering Problems," *Eur. J. Appl. Math.*, **9**, 561–588, 1998.
11. T. Hagstrom and S.I. Hariharan, "A Formulation of Asymptotic and Exact Boundary Conditions Using Local Operators," *Appl. Numer. Math.*, **27**, 403–416, 1998.
12. M.N. Guddati and J.L. Tassoulas, "Continued-Fraction Absorbing Boundary Conditions for the Wave Equation," *J. Comput. Acoust.*, **8**, 139–156, 2000.
13. D. Givoli and I. Patlashenko, "An Optimal High-Order Non-Reflecting Finite Element Scheme for Wave Scattering Problems," *Int. J. Numer. Meth. Engng.*, **53**, 2389–2411, 2002.
14. D. Givoli and B. Neta, "High-Order Non-Reflecting Boundary Conditions for Dispersive Waves," *Wave Motion*, **37** (2003), 257–271.
15. D. Givoli, B. Neta, High-Order Non-Reflecting Boundary Scheme for Time Dependent Waves, *J. Computational Physics*, **186**, (2003), 24–46.
16. R.L. Higdon, "Radiation Boundary Conditions for Dispersive Waves," *SIAM J. Numer. Anal.*, **31**, 64–100, 1994.
17. D. Givoli, B. Neta, and Igor Patlashenko, Finite Element Solution of Exterior Time Dependent Wave Problems with High-Order Boundary Treatment, *International Journal Numerical Methods in Engineering*, **58**, 1955–1983, 2003.
18. T. Hagstrom and T. Warburton, "A New Auxiliary Variable Formulation of High-Order Local Radiation Boundary Conditions: Corner Compatibility Conditions and Extensions to First Order Systems," *Wave Motion*, to appear.