Advances in Computational & Experimental Engineering & Science

An efficient BEM for numerical solution of the biharmonic boundary value problem

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Summary

This paper presents an efficient BEM for solving biharmonic equations. All boundary values including geometries are approximated by the universal high order radial basis function networks (RBFNs) rather than the usual low order interpolations. Numerical results show that the proposed BEM is considerably superior to the linear/quadratic-BEM in terms of both accuracy and convergence rate.

Introduction

Many engineering problems such as plate bending problems are governed by biharmonic equations. The numerical analysis of biharmonic boundary value problems can be accomplished by a number of techniques: FDM, FEM, BEM and other methods. The boundary element method is seen to have some advantages over the domain type solution methods. For example, the system of equations obtained by the BEM is small relative to those obtained by the FDM and FEM. Furthermore, in the case of homogeneous equations, it requires discretization only on the boundary. In a typical BEM, low order interpolations such as constant, linear or quadratic schemes are used to represent the boundary values. As a result, dense meshes are normally required in order to achieve high accuracy. Note that for the solution of integral equations of the first kind, the use of large numbers of boundary elements can lead to ill-conditioned systems of algebraic equations.

Radial basis function networks have been proven to have the property of universal approximation. Theoretically, RBFNs can represent any continuous function to a prescribed degree of accuracy using relatively low numbers of data points. However, in practice, due to the lack of theoretical determination of the network parameters, it is difficult to achieve this universal approximation. To represent a variable and its derivatives, there are two basic approaches. The first approach, namely direct RBFNs (DRBFNs), is based on the differentiation process while the second one, namely indirect RBFNs (IRBFNs), is based on the integration process. From an approximation theoretic point of view, the approximating functions are expected to be much smoother through the integration process and therefore IRBFNs can have higher approximation power than DRBFNs [1].

In the present work to deal with biharmonic equations, indirect RBFNs are introduced into the BEM scheme to represent all boundary values. Good accuracy and high rate of convergence with mesh refinement are obtained with the present method. The remainder of the paper is organized as follows. A brief review of integral equations for biharmonic equations is given in section 2. Section 3 presents the indirect RBFN approach. The proposed IRBFN-BEM method is described in section 4 and then verified through a number of

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examples including plate bending problems in section 5. Section 6 gives some concluding remarks.

Integral equations for the biharmonic boundary value problem

Consider the biharmonic equation defined on a 2D domain Ω for a function $v(x,y)$,

$$\nabla^4 v = p(x,y), \tag{1}$$

where $p(x,y)$ is a known function of position. By introducing a new variable $u = \nabla^2 v$, integral equations for the two following equations,

$$\nabla^2 u = p(x,y) \quad \text{and} \quad \nabla^4 v = p(x,y),$$

can be written as

$$C(P)u(P) + \int_\Gamma \frac{\partial G^H(P,Q)}{\partial n} u(Q) d\Gamma = \int_\Gamma G^H(P,Q) \frac{\partial u(Q)}{\partial n} d\Gamma$$
$$- \int_\Omega G^H(P,Q) p(Q) d\Omega, \tag{2}$$
$$C(P)v(P) + \int_\Gamma \frac{\partial G^H(P,Q)}{\partial n} v(Q) d\Gamma = \int_\Gamma G^H(P,Q) \frac{\partial v(Q)}{\partial n} d\Gamma$$
$$- \int_\Gamma \left(\frac{\partial G^B(P,Q)}{\partial n} u(Q) - G^B(P,Q) \frac{\partial u(Q)}{\partial n} \right) d\Gamma - \int_\Omega G^B(P,Q) p(Q) d\Omega, \tag{3}$$

where P is the source point, Q the field point, Γ the boundary of Ω, $C(P)$ the free term coefficient which is 1 if P is an internal point, $1/2$ if P is a point on the smooth boundary and $\frac{\theta}{2\pi}$ if P is a corner (θ the internal angle of the corner in radians), $\mathbf{n}$ the outwardly normal unit vector, G^H and G^B the harmonic and biharmonic fundamental solutions respectively whose forms are

$$G^H = \frac{1}{2\pi} \ln\left(\frac{1}{r}\right) \quad \text{and} \quad G^B = \frac{1}{8\pi} r^2 \left[\ln\left(\frac{1}{r}\right) + 1 \right],$$

in which $r = \|P - Q\|$.

Indirect RBFNs

Consider a function of one variable $f(s)$. The second order IRBFN scheme is employed here to represent $f(s)$. In this scheme, the second order derivative is decomposed into RBFs. The approximating function obtained is then integrated to yield expressions for the first order derivative and the original function,

$$\frac{d^2 f(s)}{ds^2} = \sum_{i=1}^{m} w^{(i)} g^{(i)}(s), \tag{4}$$
$$\frac{d f(s)}{ds} = \int \sum_{i=1}^{m} w^{(i)} g^{(i)}(s) ds + C_1 = \sum_{i=1}^{m+1} w^{(i)} H_{[1]}^{(i)}(s), \tag{5}$$
$$f(s) = \int \sum_{i=1}^{m+1} w^{(i)} H_{[1]}^{(i)} ds + C_2 = \sum_{i=1}^{m+2} w^{(i)} H_{[0]}^{(i)}(s), \tag{6}$$

where m is the number of radial basis functions (neurons), $\{g^{(i)}\}_{i=1}^{m}$ the set of RBFs, $\{w^{(i)}\}_{i=1}^{m}$ the set of network weights to be found, $\{H_{[.]}^{(i)}\}_{i=1}^{m}$ new basis functions obtained from integrating the radial basis function g. For convenience, integration constants which are unknowns here and their associated known basis functions (polynomial) on right hand sides in (4)-(6) are also denoted by the notations $w^{(i)}$ and $H_{[.]}^{(i)}$ respectively but with $i > m$. In the present work, multiquadrics is considered

$$g^{(i)}(s) = \sqrt{(s-c^{(i)})^2 + a^{(i)2}}, \tag{7}$$

in which $\{c^{(i)}\}_{i=1}^{m}$ is the set of RBF centres and $\{a^{(i)}\}_{i=1}^{m}$ is the set of RBF widths.

It is different from the previous work dealing with viscous flow problems [2], the set of network weights $\{w^{(i)}\}_{i=1}^{m+2}$ is now converted into the set of nodal variable values $\{f(s^{(i)})\}_{i=1}^{n}$ in order to make the BEM matrix square. By choosing the set of training points to be the same as the set of centres, i.e. $\{s^{(i)}\}_{i=1}^{n} = \{c^{(i)}\}_{i=1}^{m}$ with $n = m$, the evaluation of (6) at the set of training points results in

$$\begin{pmatrix} f(s^{(1)}) \\ f(s^{(2)}) \\ \vdots \\ f(s^{(n)}) \end{pmatrix} = \begin{bmatrix} H_{[0]}^{(1)}(s^{(1)}) & \cdots & H_{[0]}^{(m)}(s^{(1)}) & s^{(1)} & 1 \\ H_{[0]}^{(1)}(s^{(2)}) & \cdots & H_{[0]}^{(m)}(s^{(2)}) & s^{(2)} & 1 \\ & \cdots & & & \\ H_{[0]}^{(1)}(s^{(n)}) & \cdots & H_{[0]}^{(m)}(s^{(n)}) & s^{(n)} & 1 \end{bmatrix} \begin{pmatrix} w^{(1)} \\ w^{(2)} \\ \vdots \\ w^{(m+2)} \end{pmatrix}, \tag{8}$$

or

$$\mathbf{f} = \mathbf{H}_{[\mathbf{0}]}\mathbf{w}. \tag{9}$$

The obtained system (9) is solved using the general linear least squares. The network weights can now be expressed in terms of the function values $\{f^{(i)}\}_{i=1}^{n}$,

$$\mathbf{w} = \mathbf{H}_{[\mathbf{0}]}^{-1}\mathbf{f}. \tag{10}$$

By substituting (10) into (4)-(6), the function f and its derivatives at an arbitrary point s can be computed by

$$f(s) = \begin{bmatrix} H_{[0]}^{(1)}(s) & \cdots & H_{[0]}^{(m)}(s) & s & 1 \end{bmatrix} \mathbf{H}_{[\mathbf{0}]}^{-1} \begin{bmatrix} f^{(1)} f^{(2)} & \cdots f^{(n)} \end{bmatrix}^T, \tag{11}$$

$$\frac{df(s)}{ds} = \begin{bmatrix} H_{[1]}^{(1)}(s) & \cdots & H_{[1]}^{(m)}(s) & 1 & 0 \end{bmatrix} \mathbf{H}_{[\mathbf{0}]}^{-1} \begin{bmatrix} f^{(1)} f^{(2)} & \cdots f^{(n)} \end{bmatrix}^T, \tag{12}$$

$$\frac{d^2 f(s)}{ds^2} = \begin{bmatrix} g^{(1)}(s) & \cdots & g^{(m)}(s) & 0 & 0 \end{bmatrix} \mathbf{H}_{[\mathbf{0}]}^{-1} \begin{bmatrix} f^{(1)} f^{(2)} & \cdots f^{(n)} \end{bmatrix}^T. \tag{13}$$

The IRBFN-BEM algorithm

The procedural flow chart can be summarized as follows

- Divide the boundary into a number of segments over each of which the boundary is smooth and the prescribed boundary conditions are of the same type;

- Approximate all boundary values including geometries on each segment by IRBFNs, e.g. for geometries and the variable v,

$$x(s) = \begin{bmatrix} H_{[0]}^{(1)}(s) & \cdots & H_{[0]}^{(m)}(s) & s & 1 \end{bmatrix} \mathbf{H}_{[\mathbf{0}]}^{-1} \begin{bmatrix} x^{(1)} & x^{(2)} & \cdots & x^{(n)} \end{bmatrix}^T,$$
$$y(s) = \begin{bmatrix} H_{[0]}^{(1)}(s) & \cdots & H_{[0]}^{(m)}(s) & s & 1 \end{bmatrix} \mathbf{H}_{[\mathbf{0}]}^{-1} \begin{bmatrix} y^{(1)} & y^{(2)} & \cdots & y^{(n)} \end{bmatrix}^T,$$
$$v(s) = \begin{bmatrix} H_{[0]}^{(1)}(s) & \cdots & H_{[0]}^{(m)}(s) & s & 1 \end{bmatrix} \mathbf{H}_{[\mathbf{0}]}^{-1} \begin{bmatrix} v^{(1)} & v^{(2)} & \cdots & v^{(n)} \end{bmatrix}^T,$$
$$\frac{\partial v(s)}{\partial n} = \begin{bmatrix} H_{[0]}^{(1)}(s) & \cdots & H_{[0]}^{(m)}(s) & s & 1 \end{bmatrix} \mathbf{H}_{[\mathbf{0}]}^{-1} \begin{bmatrix} \frac{\partial v^{(1)}}{\partial n} & \frac{\partial v^{(2)}}{\partial n} & \cdots & \frac{\partial v^{(n)}}{\partial n} \end{bmatrix}^T;$$

- Substitute the IRBFNs representing the boundary values into the integral equations (IEs) (2)-(3) and then discretize the IEs;
- Impose the boundary conditions;
- Solve the system of algebraic equations obtained by Gaussian elimination;
- Evaluate the interior solution at selected internal points.

Numerical examples

In the following examples, for simplicity, the width of the ith neuron $(a^{(i)})$ is chosen to be the minimum distance from the ith centre to neighbouring centres. The accuracy of numerical solution produced by an approximation scheme is measured via the norm of relative errors of the solution as follows

$$N_e = \sqrt{\frac{\sum_{i=1}^{n_t} \left[f_0(s^{(i)}) - f(s^{(i)}) \right]^2}{\sum_{i=1}^{n_t} f_0(s^{(i)})^2}}, \tag{14}$$

where n_t is the number of test points, $s^{(i)}$ is the ith test point, f and f_0 are the calculated and exact functions respectively.

Example 1 Consider the homogeneous biharmonic equation $\nabla^4 v = 0$ on a unit square domain, subject to the boundary conditions for u and v. The exact solution is given by

$$v = \frac{1}{2}\left(x \sin x \cosh y - x \cos x \sinh y\right).$$

Ten uniform discretizations of the boundary, namely $3 \times 4, 5 \times 4, \cdots, 21 \times 4$ (number of nodes per segment $\times$ number of segments), are employed to study mesh convergence. A set of 26×4 test points is used to compute the error norm N_e for all different meshes and their results are displayed in Figure 1-a. Results of N_e obtained by the linear and quadratic BEMs are also included for comparison. The present method gives the most accurate results, followed by the quadratic-BEM and the linear-BEM. At the first four coarse meshes, the convergence rates are of $O(\bar{s}^{1.76})$, $O(\bar{s}^{2.70})$ and $O(\bar{s}^{3.48})$ for the linear, quadratic and IRBFN BEMs respectively in which $\bar{s}$ denotes the boundary node spacing.

Example 2 This problem is the same as the previous one, except that the domain of interest is a circle of radius $R=2$ and the boundary conditions are of the types v and $\partial v/\partial n$. This problem provides a good means to test and validate numerical methods. A number of uniform meshes $(15\times 2, 20\times 2, \cdots, 40\times 2)$ and a test set of 50×2 points are employed along the boundary. Again, the present method yields the best performance as shown in Figure 1-b. It should be pointed out that the linear-BEM here performs better than the quadratic-BEM. The reason could be that the boundary solution u obtained by the latter is less smooth (only C_0 continuous to be guaranteed here) than that obtained by the former, resulting in larger fluctuations in the boundary solution $\partial u/\partial n$. The solutions converge apparently as $O(\bar{s}^{1.78})$, $O(\bar{s}^{1.15})$ and $O(\bar{s}^{2.43})$ for the linear, quadratic and IRBFN BEMs respectively in which $\bar{s}$ denotes the nodal spacing.

Example 3 A simply supported square plate of $[0,200]\times[0,200]$ cm ^{2} with an uniform load q is considered here. The parameters of the problem are

$$E=2.1\times 10^6\text{kg/cm}^2,\ q=0.5\text{kg/cm}^2, h=10\text{cm},\ \nu=0.3, D=Eh^3/12(1-\nu^2).$$

Table 1 summarizes the results on the boundary obtained by the present method and also by the linear-BEM [3]. With the same mesh of 9×4 used, the present method achieves much more accurate results than the linear-BEM.

Conclusion

This paper reports a new BEM for the analysis of biharmonic problems. Unlike conventional BEMs, all boundary values in the biharmonic integral equations are represented by the universal high order RBFNs. For a better quality of approximation, all networks are constructed based on the integration process rather than the differential process. To make the BEM system of equations square, prior conversions of the sets of network weights into the sets of variable values are employed. Numerical examples show that the proposed method performs much better than the linear/quadratic BEM in terms of both solution accuracy and convergence rate.

References

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Table 1: Plate bending problem.

	$\frac{\partial v}{\partial n} \times 10^{-3}$ (error)			
Method	$(x=25, y=0)$	$(x=50, y=0)$	$(x=75, y=0)$	$(x=100, y=0)$
Analytical	-0.1146	-0.2048	-0.2613	-0.2804
IRBFN-BEM	-0.1150(0.35%)	-0.2051(0.15%)	-0.2616(0.11%)	-0.2807(0.11%)
BEM [3]	-0.126 (9.95%)	-0.222 (8.40%)	-0.276 (5.63%)	-0.298 (6.28%)
	$D\frac{\partial u}{\partial n} \times 10^{2}$ (error)			
Analytical	0.1959	0.2813	0.3243	0.3376
IRBFN-BEM	0.1961(0.10%)	0.2812(0.04%)	0.3244(0.03%)	0.3376(0.00%)
BEM [3]	0.2072(5.77%)	0.2828(0.53%)	0.3277(1.05%)	0.3394(0.53%)

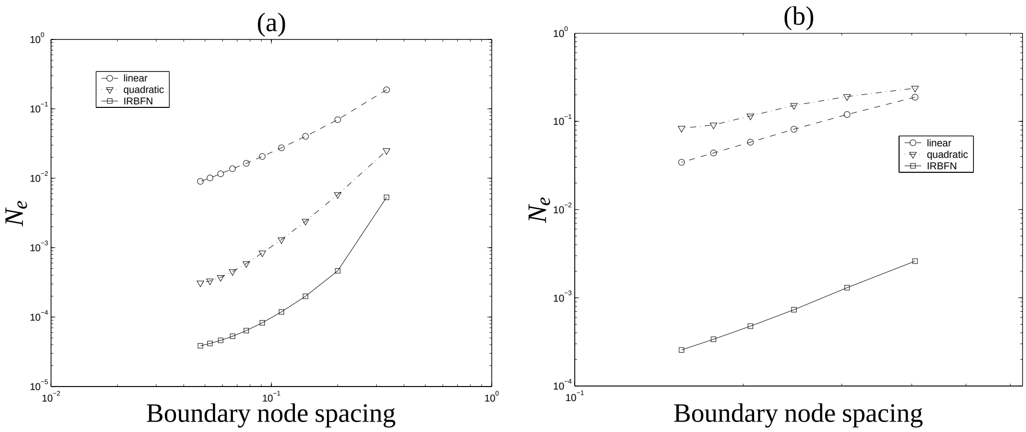

Figure 1: Results of the error norm N_e obtained by the linear, quadratic and IRBFN BEMs. The present method is superior to the linear and quadratic BEMs in terms of both solution accuracy and convergence rate.