Identification of Thermal Boundary Conditions using A Transient Temperature History

K. Momose[1], Y. Sawada[2], H. Kimoto[1]

Summary

A non-iterative approach based on adjoint formulation of conduction heat transfer is proposed to identify the thermal boundary conditions from a transient temperature history measured in a solid body. Using a numerical solution of the adjoint problem, which can be regarded as a numerical Green's function, the temperature at the measuring point can be evaluated under arbitrary thermal boundary conditions. As a result, we can inversely predict the thermal boundary conditions from the measured temperature history by assuming the step-wise profiles of thermal boundary conditions both in time and space.

Introduction

Since inverse heat conduction problems (IHCP) arise in many areas of engineering and science, a number of solution methods have been developed for the IHCP[1,2]; especially, the inverse problem of estimating the thermal boundary conditions from temperature measurements in a heat conducting body is constantly of a great interest[3]. For this kind of problem, an optimization strategy with regularization technique is often applied, in which the differences between the measured temperatures and simulated ones are iteratively minimized. In such methods, however, a large number of numerical simulations for the heat conduction field are required in the iterative optimization process; this leads to a large computational load.

In this paper, a non-iterative approach based on adjoint formulation of conduction heat transfer is proposed, the goal of which is to identify the thermal boundary conditions from a single transient temperature history measured in a heat conducting body. The first step for developing the present method is to formulate the general relationship between the thermal boundary conditions and the temperature at the measuring point. We use the weak formulation of heat conduction problem to derive the relationship. Then, by using a numerical solution of the resulting adjoint problem, which can be regarded as a numerical Green's function whose one point is fixed at the measuring point, the temperature at the measuring point can be evaluated under arbitrary thermal boundary conditions. Thus, in the inverse step, we can non-iteratively predict the thermal boundary conditions from the measured temperature history by assuming the step-wise profiles of thermal boundary conditions both in time and space. The results of computational experiments in a two-dimensional heat conduction problem are presented to demonstrate the present method.

---

[1] Graduate School of Engineering and Science, Osaka University, 560-8531, Japan
[2] Denso Corporation, Japan

Problem Formulation

Consider a finite solid body with constant material properties, and let Ω be an inner domain and Γ its boundary. Then the heat conduction equation can be written in a dimensionless form as

$$\frac{\partial T(\boldsymbol{x},t)}{\partial t} = \nabla^2 T(\boldsymbol{x},t) \tag{1}$$

where $\boldsymbol{x}$ is the space vector, t is the time, and T is the temperature.

Here, we introduce a temperature q $(\boldsymbol{x},t)$ defined as the difference from a known initial temperature distribution, and we suppose that the boundary Γ consists of the temperature-specified boundary Γ_q and the heat-flux-specified boundary Γ_q. Then, the governing equation and the initial and boundary conditions adopted in this study can be summarized as follows:

$$\frac{\partial q(\boldsymbol{x},t)}{\partial t} = \nabla^2 q(\boldsymbol{x},t) \quad \boldsymbol{x} \in \Omega, t > 0 \tag{2}$$

$$q(\boldsymbol{x},0) = 0 \quad \boldsymbol{x} \in \Omega \tag{3}$$

$$q(\boldsymbol{x},t) = q_s(\boldsymbol{x},t) \quad \boldsymbol{x} \in \Gamma_q, \quad q(\boldsymbol{x},t) \equiv \frac{\partial q}{\partial n} = q_s(\boldsymbol{x},t) \quad \boldsymbol{x} \in \Gamma_q \tag{4}$$

where q_s and q_s represent the boundary temperature and the boundary heat flux, and $\boldsymbol{n}$ denotes the outward unit vector normal to the boundary.

Under these assumptions, the problem is to identify the temperature or heat flux on part of the boundary from a transient temperature history measured at a point within the domain.

Identification Methodology

For convenience in representing the integral equations derived later, we denote Eq.(2) by using the linear operator $\mathcal{L}$, such that

$$\mathcal{L}[q(\boldsymbol{x},t)] = 0, \qquad \mathcal{L} \equiv \frac{\partial}{\partial t} - \nabla^2 \tag{5}$$

Let us now consider the weak form of Eq.(5), which can be expressed as

$$\int_0^t \int_\Omega q^* \mathcal{L}[q] \, d\Omega dt = 0 \tag{6}$$

where q^* is a test function or an adjoint temperature and t is the specific time.

Applying integration by parts to Eq.(6), we obtain the following integral equation:

$$\int_0^t \int_\Omega q \mathcal{L}^* \left[q^* \right] d\Omega dt = \int_0^t \int_\Gamma \left(q^* q - q^* q \right) d\Gamma dt - \int_\Omega \left[q^* q \right]_0^t d\Omega \tag{7}$$

where q^* is the adjoint heat flux, and $\mathcal{L}^*$ denotes the adjoint operator, which can be expressed as

$$\mathcal{L}^* = -\frac{\partial}{\partial t} - \nabla^2 \tag{8}$$

In order to eliminate the last term in the right-hand side of Eq.(7), we set the adjoint temperature at t as

$$q^*\left(\boldsymbol{x}, t\right) = 0 \quad \boldsymbol{x} \in \Omega \tag{9}$$

which can be regarded as an initial condition for the adjoint problem.

Moreover, in order to evaluate the thermal boundary condition effects on the temperature measured at x and at t, we choose the following adjoint problem:

$$\mathcal{L}^* \left[q^* \left(\boldsymbol{x}, t \right) \right] = d\left(x - \boldsymbol{x} \right) d\left(t - t \right) \tag{10}$$

$$q^*\left(\boldsymbol{x}, t\right) = 0 \quad \boldsymbol{x} \in \Gamma_q, \qquad q^*\left(\boldsymbol{x}, t\right) = 0 \quad \boldsymbol{x} \in \Gamma_q \tag{11}$$

where $d\left(\cdot\right)$ is Dirac's delta function.

Finally, substituting Eqs.(9)-(11) into Eq.(7), we obtain the following boundary integral relationship:

$$q\left(x, t\right) = \int_0^t \int_{\Gamma_q} q^*\left(\boldsymbol{x}, t - t\right) q_s\left(\boldsymbol{x}, t\right) d\Gamma dt - \int_0^t \int_{\Gamma_q} q^*\left(\boldsymbol{x}, t - t\right) q_s\left(\boldsymbol{x}, t\right) d\Gamma dt \tag{12}$$

Equation (12) indicates that if we numerically solve the adjoint equation (10) under the initial and boundary conditions in Eqs.(9) and (11), we can predict the temperature at x and at t under arbitrary thermal boundary conditions.

Let us now consider an inverse problem, in which the thermal boundary conditions should be evaluated by a temperature history measured at x. In this study, we assume that the boundary is divided into N sub-boundaries, on each of which the thermal boundary condition is uniform. We also use the following step-wise assumption in time; the thermal boundary conditions are constant over a short period of time and we can obtain M ($M \geq N$) temperatures at x over each time period.

According to Eq.(12) under the step-wise assumptions above, the temperatures measured at x over a short time period ($0 < t < t_M$) can be written in matrix form, such as

$$q = \boldsymbol{Cb} \tag{13}$$

where

$$q = \begin{Bmatrix} q(x,t_1) \\ q(x,t_2) \\ \vdots \\ q(x,t_M) \end{Bmatrix}, \qquad \boldsymbol{b} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{Bmatrix}, \qquad \boldsymbol{C} = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ C_{M1} & C_{M2} & \cdots & C_{MN} \end{bmatrix} \tag{14}$$

In the eqations above, b_n ($n = 1,2,\cdots,N$) are unknown temperatures on Γ_q or unknown heat fluxes on Γ_q, while C_{mn} ($m = 1,2,\cdots,M$, $n = 1,2,\cdots,N$) are known coefficients that can be obtained as

$$C_{mn} = \begin{cases} -\int_0^{t_m} \int_{\Gamma_n} q^*(\boldsymbol{x}, t_m - t) d\Gamma dt & \Gamma_n \subset \Gamma_q \\ +\int_0^{t_m} \int_{\Gamma_n} q^*(\boldsymbol{x}, t_m - t) d\Gamma dt & \Gamma_n \subset \Gamma_q \end{cases} \tag{15}$$

by numerically solving the adjoint problem (10) under the boundary conditions (11).

As a result, the thermal boundary conditions can be estimated from the temperature history measured at x by solving

$$\boldsymbol{b} = \left(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C}\right)^{-1} \boldsymbol{C}^{\mathrm{T}} q \tag{16}$$

in a least squares sense.

It should be noted that the solution of the adjoint problem in Eq.(10) converges rapidly in time, because it is an impulse response from a unit impulse at *x*. In other words, only the early stage of the solution is significant and the late stage is negligible. Thus, we can compute C_{mn} in Eq.(15) at arbitrary time t_m using a truncated numerical solution.

Results of Computational Experiments

In order to demonstrate the present method, we carried out two computational experiments in a two-dimensional square body as shown in Fig.1, in which the heat flux on the bottom surface should be identified, while the temperatures on the top and side surfaces are known.

In both two computational experiments, the measured temperature histories at *x* were simulated numerically by the finite difference method. The adjoint problem (10) under the thermal boundary conditions (11), which is common to both cases in Fig.1, was also calculated by the finite difference method.

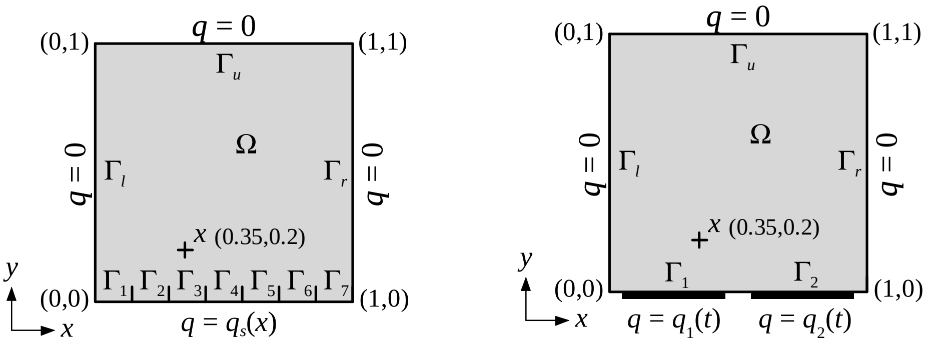

Fig.1 Identification models in a two-dimensional square body

The purpose of the first computational experiment (Case 1 in Fig.1) is to identify the heat flux distribution on the bottom surface, which is imposed at t = 0 and is maintained at constant value in time, such that

$$q_s(x)=\begin{cases} 0 & x<0.1 \quad t\geq 0 \\ \sin\dfrac{x-0.1}{0.8}p & 0.1\leq x\leq 0.9 \quad t\geq 0 \\ 0 & x>0.9 \quad t\geq 0 \end{cases}$$

In this example, the bottom surface is divided into seven sub-boundaries (N = 7) as shown in Fig.1(a), and also seven measured temperatures (M = 7) are used as indicated by circle plots in Fig.2. The identification result is shown in Fig.3; the agreement of predicted heat flux distribution with the exact one can be seen to be quite good.

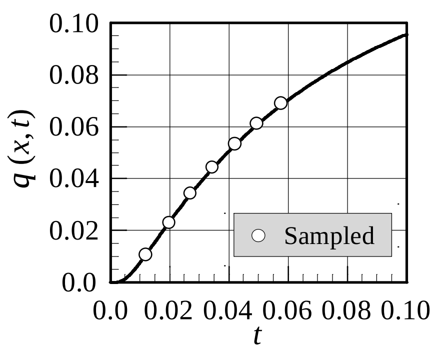

Fig.2 Measured temperature history in Case 1

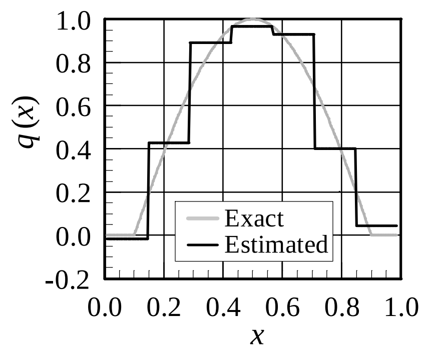

Fig.3 Identification result in Case 1

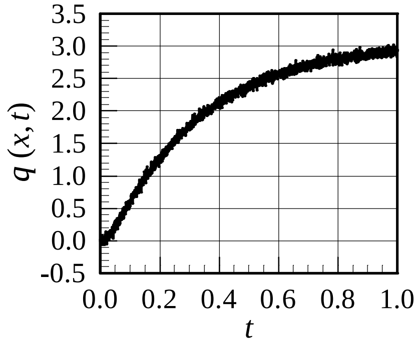

Fig.4 Measured temperature history in Case 2

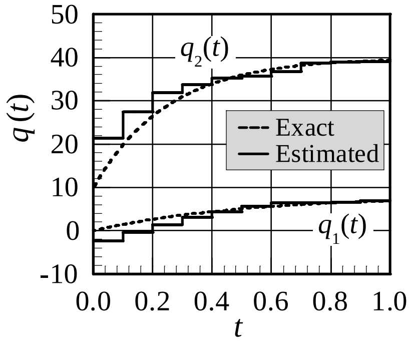

Fig.5 Identification result in Case 2

In the second computational experiment (Case 2 in Fig.1), we attempted to identify time-dependent heat fluxes imposed on the bottom surface as shown in Fig.1(b), in which the heat fluxes are assumed to be

$$q_1(t) = 8\left(1 - e^{-2t}\right) \quad \text{on } \Gamma_1, \quad q_2(t) = 30\left(1 - e^{-4t}\right) + 10 \quad \text{on } \Gamma_2$$

Fig.4 indicates the temperature history at x with a random noise of 2% of the maximum temperature. As mentioned in the previous section, we assume the step-wise profiles for both the heat fluxes $q_1(t)$ and $q_2(t)$; the period of time is 0.1 ($t_M = 0.1$) and the number of measured temperatures in each period is 250 ($M = 250$). As shown in Fig.5, the time-dependent heat fluxes can also be estimated satisfactorily by the present method.

Concluding Remarks

In this study, we propose a non-iterative method to identify the thermal boundary conditions from a temperature history measured in a solid body. The main features of the present method can be summarized as follows: Introducing an adjoint problem and its associated Green's function, the relationship between the thermal boundary conditions and the measured temperature history can directly be obtained. Using the numerical Green's function and assuming the step-wise profiles both in time and space, we can determine the thermal boundary conditions in the least squares sense.

Reference

1 Beck, J. V., Blackwell, B. and St. Clair, C. R. (1985): *Inverse Heat Conduction: Ill-Posed Problems*, Wiley Interscience.

2 Alifanov, O. M. (1994): *Inverse Heat Transfer Problems*, Springer-Verlag.

3 Edited by Orlande, H. R. B. (2002): *Proc. 4th International Cenference on Inverse Problem in Engineering: Theory and Practice*, Vol. II.