Stabilization effect of a rotating magnetic field on the flow of a conducting liquid in a cylindrical container

Jānis Priede[1], Gunter Gerbeth[2]

Summary

This study presents a numerical stability analysis of a flow of electrically conducting liquid driven by a rotating magnetic field in a cylindrical container. The aim of the work is to investigate the previously often neglected effect of the strength of the rotating magnetic field on the stability of the flow driven by the field. Linear hydrodynamic stability analysis has been carried out by Chebyshev-tau and Galerkin spectral numerical methods. We find that the strength of rotating magnetic field has a stabilizing effect on the flow. The obtained results may be of practical relevance for certain semiconductor growth technologies from the melt.

Introduction

A rotating magnetic field (RMF) is usually generated by a system of multiphase alternating currents, like in a stator of an AC electromotor, and it has a certain spatial pattern that rotates in time about some axis. When an electrically conducting body is placed in such a RMF eddy currents are induced in the body. The induced currents interact with the applied magnetic field and generate the electromagnetic torque trying to entrain the body with the field rotation in order to reduce the variation of the magnetic flux through the body. Such inductors of RMF are used to drive not only the rotor of AC motors, but they also have found application in material processing technologies ranging from metallurgical to semiconductor crystal growth processes where a conducting liquid like a molten metal or a semiconductor melt plays the role of a rotor. In metallurgy RMF's are usually applied for stirring of molten metals. From crystal growth it is known that a relatively weak RMF can significantly improve the quality of the grown material, e.g., by reducing the segregation striation in the crystal caused by unstable thermal convection. On the other hand, a too strong RMF can deteriorate the crystal quality due to hydrodynamic instabilities in the melt. The objective of this work is to investigate numerically the possible stabilization of a flow of an electrically conducting liquid driven by a RMF in a cylindrical container by means of the RMF itself. This subject presents an interest for single crystal growth applications from the melt and, more generally, to the field of hydrodynamic stability.

The study is focused on the direct stabilizing effect of the RMF itself, originally introduced by Priede & Gelfgat [1] and later employed by Moessner & Gerbeth [2]

[1]Institute of Physics, University of Latvia , LV-2169 Salaspils, Latvia
[2]Forschungszentrum Rossendorf, P.O. Box 510119, D-01314 Dresden, Germany

in numerical simulations. Till now most of the studies have considered the action of the RMF on the liquid like on a solid body, reducing it only to the creation of a torque. In this low-induction and low-frequency approximation the effect of the RMF is defined only by the magnitude of the driving azimuthal force. In other words, the related Taylor number is the only determining parameter of the problem, besides the aspect ratio of the container. Here we take into account that the Ekman pumping causing the liquid to circulate in the meridional plane, forces the liquid to cross RMF flux lines so giving rise to an additional electromagnetic braking force. Thus, we still keep the low-frequency approximation but we take into account the field strength defined by the Hartmann number of the RMF as a second independent parameter besides the magnetic Taylor number.

Problem formulation

Consider a liquid with electric conductivity σ, kinematic viscosity ν and density ρ filled in a cylindrical container with radius R and half-height H. The liquid is subject to a uniform RMF whose free-space induction distribution can be written in cylindrical coordinates as:

$$\vec{B}(\vec{r},t) = B_0\left(\vec{e}_r \sin(\phi - \omega t) + \vec{e}_\phi \cos(\phi - \omega t)\right),$$

where B_0 is the characteristic induction and ω is the angular frequency of field rotation. The current induced in the liquid is governed by Ohm's law for moving medium $\vec{j} = \sigma\left(\vec{E} + \vec{v}\times\vec{B}\right)$, where $\vec{v}$ is the liquid velocity and $\vec{E}$ is the strength of the induced electric field which is governed by the first Maxwell equation $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$. In order to find the induced electric field it is useful to proceed to the frame of reference rotating at the angular frequency $\vec{\omega} = \vec{e}_z\omega$ together with the field where the flow velocity is $\vec{v}' = \vec{v} - \vec{\omega}\times\vec{r}$ but the magnetic field is steady. Thus the first Maxwell equation governing the strength of the induced field $\vec{E}'$ in this frame of reference takes the simple form $\vec{\nabla}\times\vec{E}' = 0$ implying that $\vec{E}' = -\vec{\nabla}\Phi$, where Φ is the electrostatic potential. Then Ohm's law in the rotating frame of reference takes the form $\vec{j} = \sigma\left(\vec{E}' + \vec{v}'\times\vec{B}\right) = \sigma\left(-\vec{\nabla}\Phi + (\vec{v} - \vec{\omega}\times\vec{r})\times\vec{B}\right)$. Charge conservation $\vec{\nabla}\cdot\vec{j} = 0$ leads to the following equation for the scalar potential: $\vec{\nabla}^2\Phi = \vec{B}\cdot\vec{\nabla}\times\vec{v}$. In obtaining the above equation we have employed the low-frequency approximation implying that there is no significant skin-effect $\mu_0\sigma\omega R^2 \preceq 1$.

Further it is advantageous to represent both the RMF and the electrostatic potential in a complex form $\vec{B}(\vec{r},t) = \Re\left[\vec{B}_0 e^{i(\phi-\omega t)}\right]$, $\quad \Phi(\vec{r},t) = \Re\left[\Phi_0 e^{i(\phi-\omega t)}\right]$ where i is the imaginary unit and $\vec{B}_0 = B_0\left(\vec{e}_\phi - i\vec{e}_r\right)$ and Φ_0 are complex amplitudes of the magnetic field and the electrostatic potential, respectively. Then the equation for the latter takes the form:

$$\left(\vec{\nabla}^2 - \frac{1}{r^2}\right)\Phi_0 = B_0\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} + i\frac{\partial v_\phi}{\partial z}\right). \tag{1}$$

The electromagnetic force $\vec{f} = \vec{j}\times\vec{B}$ being a product of two quantities alternating in time with frequency ω consists of two parts: a steady and an oscillating with frequency 2ω one. As usual, the frequency ω is assumed to be sufficiently high so that the inertia of the fluid precludes any significant fluid flow in response to the oscillating part of the force which is thus henceforth neglected. In the following we need only the azimuthal components of the force and its curl:

$$\langle f_\phi\rangle = \frac{1}{2}\sigma B_0\left[\frac{\partial\Phi_0^I}{\partial z} + B_0\left(r\omega - v_\phi\right)\right], \qquad \left\langle\vec{\nabla}\times\vec{f}\right\rangle_\phi = \frac{1}{2}\sigma B_0^2\frac{\partial v_z}{\partial r}, \tag{2}$$

where $\Phi_0^I = \Im[\Phi_0]$. Finally we use the low-induction approximation implying that the azimuthal velocity of liquid rotation is negligible relative to the velocity of rotation of the field: $v_\phi \ll r\omega$. Thus we neglect v_ϕ in Eq. (2) for the azimuthal force as well as in Eq. (1) for the electrostatic potential. The RMF has only a braking effect on the meridional flow which is determined solely by the strength but not by the frequency of the field. Thus, contrary to the azimuthal driving force the meridional force is coupled to the meridional fluid flow. The boundary conditions for the electrostatic potential at the insulating container wall S follow from $\vec{n}\cdot\vec{j}_0\Big|_s =$ 0: $\frac{\partial\Phi_0}{\partial n}\Big|_s = i(\vec{e}_z\cdot\vec{n})r\omega B_0|_s$, where $\vec{n}$ is the surface normal drawn outward to the volume.

In the following we nondimensionalize all quantities by using R, R^2/ν, ν/R, and $\nu\omega B_0$ as the length, time, velocity, and electrostatic potential scales, respectively.

The fluid flow is governed by the Navier-Stokes equation with the electromagnetic body force defined by Eq. (2) and the incompressibility constraint. Since the time-averaged electromagnetic force is axisymmetric we first search for an axisymmetric base flow whose stability is to be analyzed. Therefore we divide the velocity field as $\vec{v} = v\vec{e}_\phi + \vec{u}$, where v and $\vec{u}$ are the azimuthal and meridional flow components, respectively. To satisfy the incompressibility constraint $\vec{\nabla}\cdot\vec{u} = 0$ we introduce a stream function Ψ such that $\vec{u} = \vec{\nabla}\times(\vec{e}_\phi\Psi) = -\frac{\vec{e}_\phi}{r}\times\vec{\nabla}(r\Psi)$. Then taking the azimuthal components of the Navier-Stokes equation and of its curl we end up with the dimensionless equations:

$$\frac{\partial v}{\partial t} + \frac{1}{r}(\vec{u}\cdot\vec{\nabla})(rv) = \Delta_\phi v + Tm\langle f_\phi \rangle, \tag{3}$$

$$\frac{\partial \zeta}{\partial t} + r(\vec{u}\cdot\vec{\nabla})\left(\frac{\zeta}{r}\right) - \frac{1}{r}\frac{\partial v^2}{\partial z} = \Delta_\phi \zeta + Ha^2 \frac{\partial u_z}{\partial r} \tag{4}$$

where $\Delta_\phi = \left(\vec{\nabla}^2 - \frac{1}{r^2}\right)$; and $\zeta = \left(\vec{e}_\phi \cdot \vec{\nabla}\times\vec{u}\right) = -\Delta_\phi \Psi$; $Tm = \frac{B_0^2 R^4 \omega\sigma}{2\rho\nu^2}$ is the magnetic Taylor number characterizing the strength of the azimuthal driving force; $Ha = B_0 R\sqrt{\frac{\sigma}{2\rho\nu}}$ is the Hartmann number characterizing the braking effect of the RMF on the meridional flow. The azimuthal component of the time-averaged electromagnetic force in the low-induction approximation is $\langle f_\phi \rangle = r - \frac{\partial \Phi_0^I}{\partial z}$. The corresponding Eq. (1) for Φ_0^I takes the form $\Delta_\phi \Phi_0^I = 0$. At the container wall S there are no-slip and impermeability conditions: $v|_s = \Psi|_s = \left.\frac{\partial \Psi}{\partial n}\right|_s = 0$. The boundary condition for Φ_0^I takes the form: $\left.\frac{\partial \Phi_0^I}{\partial n}\right|_s = n_z r$. Axial symmetry implies that v and Ψ as well as ζ must be odd functions of r and thus their power series expansion can contain only odd powers of r. This constraint is taken into account in the following numerical approximation of the problem.

Numerical method

The problem is solved by a Galerkin-Chebyshev spectral numerical method. We first transform the axial coordinate as $z \to zA$, where $A = H/R$ is the aspect ratio, in order to get the transformed z into the range $[-1;1]$ and search both the azimuthal velocity velocity and the stream function as:

$$v^{(s)}(r,z,t) = \sum_{m=0}^{M}\sum_{n=0}^{N} v_{m,n}(t) v_{2m+1}(r) v_{2n+s}(z), \tag{5}$$

$$\psi^{(\bar{s})}(r,z,t) = \sum_{m=0}^{M}\sum_{n=0}^{N} \psi_{m,n}(t) \psi_{2m+1}(r) \psi_{2n+\bar{s}}(z), \tag{6}$$

where the Galerkin basis satisfying the boundary and symmetry conditions are defined in terms of Chebyshev polynomials $T_k(x)$ as:

$$\begin{aligned} v_k(x) &= T_k(x) - T_{k+2}(x), \\ \psi_k(x) &= (k+3)T_k(x) - 2(k+2)T_{k+2}(x) + (k+1)T_{k+4}(x). \end{aligned}$$

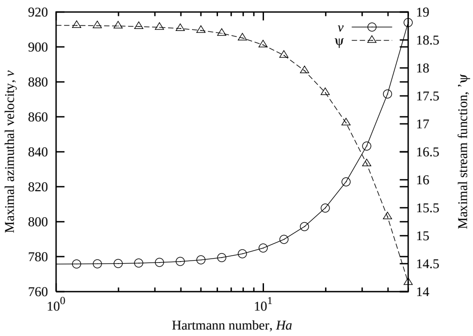

Figure 1: Maximal dimensionless values of the azimuthal velocity and stream function versus Hartmann number for magnetic Taylor number $Tm = 10^5$ and aspect ratio $A = 1$.

The subscripts s and $\bar{s} = 1 - s$ are used to denote the symmetry of the solution with respect to the midplane where $s = 0$ and $s = 1$ correspond to mirror-symmetric and -antisymmetric solutions, respectively. Note that in this notation the base flow has a $s = 0$ symmetry that ensures separation of vertically symmetric and antisymmetric perturbations in the following. Upon substituting series (5,6) into Eqs. (3,4) and projecting both equations onto basis functions $v_{2m+1}(r)v_{2n+s}(z)$ and $\psi_{2m+1}(r)\psi_{2n+\bar{s}}$ for $m = 0, 1..., M$ and $n = 0, 1, ..., N$, respectively, we obtain an algebraic system of equations for the unknown coefficients $v_{m,n}$ and $\psi_{m,n}$.

For aspect ratio $A = 1$ and no braking effect $(Ha = 0)$ the method was found to yield the instability threshold, i.e., the critical magnetic Taylor number and critical frequency of the instability, with an accuracy of about seven digits when 24 and 44 basis functions were used in radial and axial directions, respectively [3].

Numerical results

The results plotted in Fig.1 show that an increase of the braking effect for increasing Hartmann number reduces the meridional flow as expected. On the other hand, it leads to an increase of the rate of azimuthal flow for a fixed magnetic Taylor number. The latter effect of acceleration of the azimuthal flow with suppression of the meridional one results from the reduced advection of the angular momentum of the primary azimuthal flow by the reduced secondary one that actually determines the magnitude of the flow in the strongly nonlinear regime under consideration. Regarding flow stability an increase of the Hartmann number results in the stabilization of mirror-symmetric perturbations accompanied first by a slight destabilization

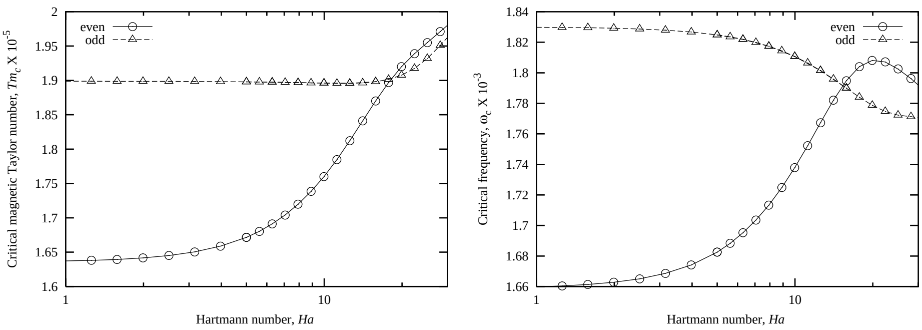

Figure 2: Critical magnetic Taylor number (*left*) and critical frequency (*right*) versus the Hartmann number for axially symmetric (*even*) and antisymmetric (*odd*) instability modes in a cylindrical container of aspect ratio $A = 1$.

of the anti-symmetric mode as illustrated on Fig 2. Nevertheless for weak magnetic fields the anti-symmetric mode remains more stable than the symmetric one. A noticeable stabilization of the anti-symmetric modes starts at $Ha \approx 20$ whereas for the symmetric part it could be attributed to start at $Ha \approx 5$. With increase of the Hartmann number the instability switches from the symmetric to the antisymmetric mode which determines the instability at higher Hartman numbers. In this way the linear stability of the flow can be increased with respect to axisymmetric perturbations from the critical magnetic Taylor number $Tm_c \approx 1.6 \times 10^5$ at weak magnetic fields $(Ha \ll 1)$ to about $Tm_c \approx 2 \times 10^5$ at $Ha = 30$.

An advantage of this stabilization method for, e.g., crystal growth applications is that it can be implemented using the inductor of the RMF alone without installing an additional magnetic system for a superimposed DC magnetic field. On the other hand, a limitation of the considered stabilization is the often limited maximal attainable strength of the RMF.

Reference

1. Priede, J. and Gelfgat, Yu. M. (1996) "Mathematical model of the mean electromagnetic force induced by a RMF in a liquid column of finite length," *Magnetohydrodynamics* Vol 32., pp. 249-256.
2. Moessner, R. and Gerbeth, G. (1999) "Buoyant melt flows under the influence of steady and rotating magnetic fields," *J. Cryst. Growth* Vol 197, pp. 341-354.
3. Grants, I. and Gerbeth, G. (2001) "Stability of axially symmetric flow driven by a RMF in a cylindrical cavity," *J. Fluid Mech.* Vol. 431, pp. 407-426.