Antiplane Stress Analysis of a Non-homogeneous Material Wedge

Yi-Liang Ou[1] and Ching-Hwei Chue[2]

Summary

The wedge structure in non-homogeneous material with specified functional property is studied by using Mellin transform. The stress singularities depend on the prescribed boundary conditions, the wedge angle and the non-homogeneity parameter. The complete expressions of the stress components and displacement can be derived.

Introduction

The stress singularity is an index to judge the extent of stress concentration near the tip of a wedge shape structure. Higher singularity order induces more possibility of failure

The singularity order of crack in non-homogeneous material has been widely discussed by several literatures [1, 2]. They found that it remains the same as those for homogeneous cases. They assumed that the material properties are in exponential forms. In this paper, we adopt the r-type material form, which has been used to study the singular order of a crack subjected to anti-plane shear [3].

Formulations of the Problem

Consider a non-homogeneous material wedge with wedge angle 2α shown in

1 Graduate Student.
2 Corresponding Author; Professor, Dept. M.E., National Cheng Kung University, Tainan, Taiwan, R.O.C.

Fig.1. The shear modulus is assumed in the form $\mu(r)=\mu_0\,[1+(r/r_c)^{\beta}]$, where r is the radial distance measured from the tip of the wedge and r_c is a characteristic length. The non-homogeneity of the material is characterized by the parameter β, which lies between –1 and 1.Positive β means that the rigidity increases from the wedge tip whereas negative one represents the decreasing rigidity from apex.

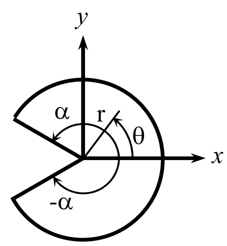


Fig.1 The geometry of a wedge structure

The stress-displacement relations and the equilibrium equation for antiplane problem are as follows:

$$\tau_{\theta z}=\frac{\mu(r)}{r}\frac{\partial w}{\partial\theta},\ \ \tau_{rz}=\mu(r)\frac{\partial w}{\partial r} \tag{1}$$

$$\frac{\partial}{\partial r}(r\tau_{rz})+\frac{\partial}{\partial\theta}(\tau_{\theta z})=0 \tag{2}$$

Substituting the form of material property into Eq.(2), the result becomes:

$$\left[\mu_0\frac{\partial w}{\partial r}+\mu_0 r\frac{\partial^2 w}{\partial r^2}+\mu_0 r^{-1}\frac{\partial^2 w}{\partial\theta^2}\right] + \left[(\beta+1)\mu_0(\frac{r}{r_c})^{\beta}\frac{\partial w}{\partial r}+\mu_0(\frac{r}{r_c})^{\beta} r\frac{\partial^2 w}{\partial r^2}+\mu_0(\frac{r}{r_c})^{\beta} r^{-1}\frac{\partial^2 w}{\partial\theta^2}\right]=0 \tag{3}$$

The first and second terms in Eq.(3) correspond to the case of a homogeneous

($\mu = \mu_0$) and a non-homogeneous ($\mu(r) = \mu_0(r/r_c)^{\beta}$) wedge, respectively. Since Ma and Hour [4] has discussed the homogeneous case, we only study the effect comes from the material non-homogeneity.

We define $\hat{w}$, $\hat{\tau}_{\theta z}$ and $\hat{\tau}_{rz}$ as the Mellin transformed function of w, $r^{1-\beta}\tau_{\theta z}$ and $r^{1-\beta}\tau_{rz}$ in the s-domain as

$$\hat{w}(s,\theta) = \int_0^{\infty} w(r,\theta) r^{s-1} dr \tag{4}$$

$$\hat{\tau}_{\theta z}(s,\theta) = (\frac{\mu_0}{r_c^{\ \beta}}) \frac{d}{d\theta} \hat{w}(s,\theta) \tag{5}$$

$$\hat{\tau}_{rz}(s,\theta) = -(\frac{\mu_0}{r_c^{\ \beta}}) s\hat{w}(s,\theta) \tag{6}$$

The equilibrium equation in the s-domain and its general solution are

$$\frac{d^2}{d\theta^2} \hat{w}(s,\theta) + (s^2 - \beta s)\hat{w}(s,\theta) = 0 \tag{7}$$

$$\hat{w}(s,\theta) = A(s)\cos(\theta\sqrt{s^2 - \beta s}) + B(s)\sin(\theta\sqrt{s^2 - \beta s}) \tag{8}$$

where A(s) and B(s) are unknown functions which may be determined from the prescribed boundary conditions. Hereafter we will discuss three different types of conditions: (i) traction-traction , $\tau_{\theta z}(r,\alpha) = T_0\delta(r-r_0)$ and $\tau_{\theta z}(r,-\alpha) = T_0\delta(r-r_0)$; (ii)displacement-displacement $w(r,\alpha)=0$, $w(r,-\alpha) = w_0(r/R)$ for 0□*r*□*R*; and (iii) traction-displacement $\tau_{\theta z}(r,\alpha) = \tau_0$ for 0<*r*□*R* and $w(r,-\alpha)=0$. The constants T_0 , w_0 and τ_0 represent the magnitudes of the shear force, displacement and shear stress, respectively. δ is the Dirac-Delta function. r_0 and R are characteristic lengths.

After substituting these specific boundary conditions to the constitutive and equilibrium equtions, the unknown coefficients A(s) and B(s) might then be obtained. Therefore the transformed terms of displacement and stresses, $\hat{w}$, $\hat{\tau}_{\theta z}$ and $\hat{\tau}_{rz}$, can be expressed explicitly. According to the theorem of inverse Mellin transform, the stress component can be obtained as following:

$$\tau_{\theta z}(r,\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{\tau}_{\theta z}(s,\theta) r^{\beta-(1+s)} ds \tag{9}$$

By applying the residue theorem, the stress $\tau_{\theta z}$ of the traction boundary condition can be written as

$$\tau_{\theta z(r\le r_0)} = T_0 r_0^{-1} \sum_{k=1}^{\infty} \frac{\pi}{\alpha^2} \left(\frac{(-1)^{k+1}(2k-1)\cos\left(\frac{2k-1}{2\alpha}\pi\theta\right)}{\sqrt{\beta^2 + ((2k-1)\frac{\pi}{\alpha})^2}} \right) (\frac{r_0}{r})^{1+s_k^- -\beta} \tag{10}$$

The stress τ_{rz} and the displacement w can be obtained in the same way.

Results and Discussions

From the expression form in Eq.(10), we see that as $r \to 0$, i.e. near the wedge tip, the stress magnitude is proportional to $r^{-(1+s_k^- -\beta)}$. Here s_k^- represents the negative pole of the infinite integral in Eq.(9). If the order lies between –1 and 0, the stress may go to infinity.

For traction-traction case (i), the variations of singular order with wedge angle at different non-homogeneous parameter β are plotted in Fig.2. The strength of stress singularity for negative β is stronger than that for positive β. For homogeneous material wedge ($\beta = 0$), the stress field is not singular when α □ $\pi/2$ [4]. In our case, the stress field becomes singular for $\alpha < \pi/2$ when β is negative. On the contrary, we may find the wedge angle $\alpha > \pi/2$ such that the stress field is nonsingular when β is positive. In the case of a crack in a homogeneous material (i.e. $\beta = 0$, $\alpha = 180^0$), it becomes the conventional square root singularity as in fracture mechanics.

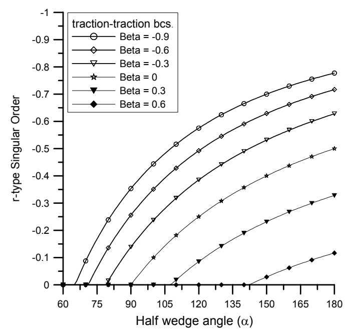


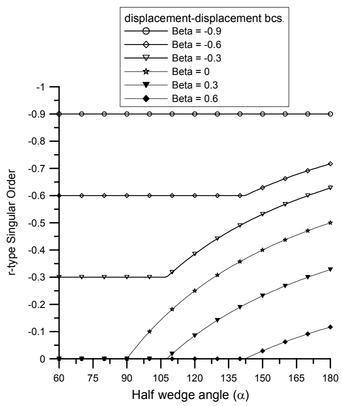
Fig.2 The variations of singular order with wedge angle at different non-homogeneous parameter β (case (i)).

Fig. 3 and 4 plot the variations of singularity order for cases (ii) and (iii), respectively. In case (ii), the variation tendency is similar to case (i) when β is positive. However, the strength of singularity becomes very strong and will dominate the stress field when β is negative. This phenomenon can be explained by observing the first term in the expression of stress $\tau_{\theta z}$:

$$\begin{aligned}\tau_{\theta z\,(r\le R)} &= -w_0\left(\frac{\mu_0}{r_c^{\ \beta}}\right)\frac{\sqrt{(1+\beta)}\cos\left((\alpha-\theta)\sqrt{(1+\beta)}\right)}{\sin\left(2\alpha\sqrt{(1+\beta)}\right)}\left(\frac{1}{r}\right)^{-\beta} \\ &+ w_0\left(\frac{\mu_0}{r_c^{\ \beta}}\right)R^{\beta}\sum_{k=1}^{\infty}\frac{\pi^2}{2\alpha^3}\frac{(-1)^k k^2\cos\left((\alpha-\theta)\dfrac{k\pi}{2\alpha}\right)}{\sqrt{\beta^2+(\frac{k\pi}{\alpha})^2}\left[2+\beta-\sqrt{\beta^2+(\frac{k\pi}{\alpha})^2}\right]}\left(\frac{R}{r}\right)^{1+s_k^{\,-}-\beta}\end{aligned} \qquad (11)$$

The variations of singularity order for case (iii) are very similar to the case (i). For homogeneous material wedge ($\beta = 0$), the stress field is not singular when α

□ $\pi/4$ [4]. In our case, the stress field becomes singular for $\alpha < \pi/4$ when β is negative and may be nonsingular for $\alpha > \pi/4$ when β is positive. For $\beta = 0$ and $\alpha = 180^0$, the singularity order is –0.75, which matches the result of [4].



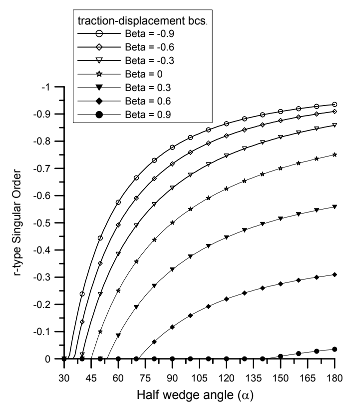


Fig.3 The variation of singular order with wedge angle at different non-homogeneous parameter β (case (ii)).

Fig.4 The variation of singular order with wedge angle at different non-homogeneous parameter β (case (iii)).

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