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An elasticity solution for functionally graded beams using stress function

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Abstract *An elasticity solution is obtained for a functionally graded beam subjected to transverse loads by using stress function method. The young modulus of the beam is assumed to vary exponentially through the thickness of the beam, and the Poisson ratio is held constant. By these assumptions the elastic coefficients of the problem vary exponentially that allows an exact solution for the elasticity problems By this method, all transverse loading that its intensity is zero at the two ends of the beam can be treated..*

Introduction

The concept of functinally graded materials (FGMs), i.e. composites with smoothly varrying constitutive properties, was first suggested by Niino and coworkers at the National Aerospace laboratory in Japan ([1, 2]). The Original idea was to manufacture super heat resistant components for use in the engines and airframe of supersonic plane, combining the heat resistance of ceramics with the structural properties of metals, an optimal non-homogeneous distribution of the second phase ceramic material was to be employed in this context.

On the analysis of non-homogeneous grade structures, there had been relatively little investigation until recently. With the advent of FGMs though there has been a renewed in inhomogeneous elasticity. Although FGMs are highly heterogeneous, it will be useful to idealize them as continua with properties changing smoothly with respect to the spatial coordinates. This will enable obtaining close-form solutions to some fundamental solid mechanics problems. Aboudi et. al.[3, 4], developed a higher order micromechanical theory for FGMs (HOTFGM) that explicitly couples the local and global effects. Later the theory was extended to free-edge problems by Aboudi and Pindera [6]. Pindera and Dunn [7] evaluated the higher order theory by performing a detailed finite element analysis of the FGM. They found that the HOTFGM results agreed well with the FE results.

There are other approximations that can be used to model the variation of properties in a FGM. One such variation is the exponential variation, where the elastic constants vary according to formulas of the type: $c=c^0e^k$. Many researchers have found this functional form of property variation to be convenient in solving elasticity problems [8]. For example, Sankar [9] obtained an elasticity solution for this type of functionally graded beams. In this paper we analyze a FGM beam as considered in [9] by a different method, using stress function. Also, we solve more General type of loading than [12].

Elasticity solution

Consider the FGM beam shown in Fig.1. It must be noted that the x-axis is along the bottom of the beam. The length of the beam is L and thickness is h. The width of the beam in the y-direction is taken as unity. The boundary conditions are like of a simply supported beam that are explained latter. The bottom Surface of the beam (z=0) is

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subjected to normal traction with intensity equal to zero at the two ends. So, the Fourier series of dhghg this traction can be written as:

$$\sigma_{zz} = -p(x) = -\sum_n p_n \sin \eta_n x \tag{1}$$

where $\eta = \frac{n\pi}{L}$, n=1,2,3,… (2)

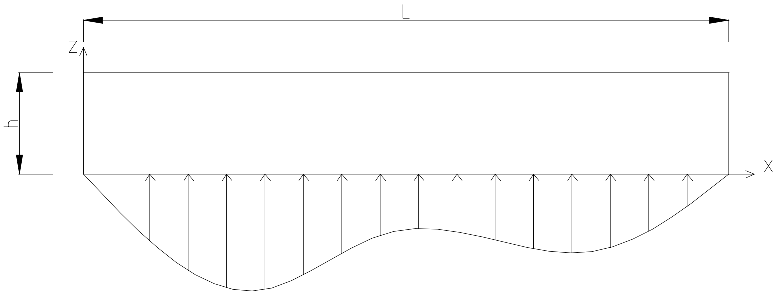

Fig.1. A FG Beam subjected to a arbitary loading that its intensity is zero at two ends.

The upper surface, z=h is completely free of tractions, and the lower surface is free of shear tractions. In this paper the problem is more general than [12] that n is not necessary to be odd, and the loading can be symmetric/antisymmetric or any combination of them about the center of the beam. The loading given by Eq.(1) is of practical significance because any arbitary normal loading that its intensity is zero at x=0 and x=L, can be expressed as a Fourier series involving terms of the type $p_n \sin \eta_n x$.

The differential equations of equilibrium are:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0, \quad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \tag{3}$$

For satisfying these equilibrium equations it is sufficient that we have a arbitary stress function ϕ and then the stresses are defined as:

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial z^2}, \quad \sigma_{zz} = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xz} = \frac{\partial^2 \phi}{\partial x \partial z} \tag{4}$$

Now, we express all quantities as the sum of fractions which each fraction is related to a specifiec n. For example we have

$$\sigma_{xx} = \sum_n \sigma_{xx,n} \tag{5}$$

From the condition $\sigma_{zz,n}(x,0) = -p_n \sin \eta_n x$ we conclude that:

$$\frac{\partial^2 \phi_n}{\partial x^2}(x,0) = -p_n \sin \eta_n x \tag{6}$$

now by integrating, we have

$$\phi_n(x,z) = \frac{f_n(z)}{\eta_n^2} \sin \eta_n x + A_n(z)x + B_n(z) + S_n(z) g_n(x) \tag{7}$$

where for satisfying (6), some restrictions must be considered for $f_n(0)$ and $S_n(0)$.

We must also have $\sigma_{zz} = 0$ at the surface z=h, thus from relations (4) and (7) it is concluded that

$$g_n(x) = -\frac{f_n(h)}{\eta_n^2 S_n(h)} \sin \eta_n x \tag{8}$$

Now, we can rewrite relation (7) as:

$$\phi_n(x,z) = \frac{\bar{f}_n(z)}{\eta_n^2} \sin \eta_n x + A_n(z)x + B_n(z) \tag{9}$$

where for satisfying (6) we must have

$$\bar{f}_n(0) = p_n\,, \quad \bar{f}_n(h) = 0 \tag{10}$$

From (9) and (4) it can be obtained:

$$\sigma_{xx,n} = \frac{\bar{f}_n^{''}(z)}{\eta_n^2} \sin \eta_n x + A_n^{''}(z)x + B_n^{''}(z) \tag{11-a}$$

$$\tau_{xz,n} = -\bar{f}_n^{'}(z)\frac{\cos \eta_n x}{\eta_n} - A_n^{'}(z) \tag{11-b}$$

$$\sigma_{zz,n} = -\bar{f}_n(z) \sin \eta_n x \tag{11-c}$$

Although each stress field that is resulted from stress function defined at (9), satisfies equilbrium equations, in addition it is necessary that the resulting stress components produce compatible strain field. In two dimensional problems only compatiblilty equation is:

$$\frac{\partial^2 \varepsilon_{xx}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} - 2\frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z} = 0 \tag{12}$$

We assume state of plain strain, thus:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ \varepsilon_{xz} \end{Bmatrix} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \tau_{xz} \end{Bmatrix} \tag{13}$$

Also, the moduli is assumed to vary exponentially as

$$E = E(z) = E_0 e^{\lambda z} \tag{14}$$

according to relations (13), (14), (12) and (4) it can be obtained

$$\Delta^4 \phi_n - 2\lambda \frac{\partial^3 \phi_n}{\partial z^3} - 2\lambda \frac{\partial^3 \phi_n}{\partial x^2 \partial z} + \lambda^2 \frac{\theta^2 \varphi_n}{\partial z^2} - \lambda^2 \frac{\nu}{1-\nu} \frac{\partial^2 \phi_n}{\partial x^2} = 0 \tag{15}$$

Now by inserting ϕ_n from (9) in (15), it can be resulted

$$\begin{gathered} \left[f_n^{''''}(z) - 2\lambda f_n^{'''}(z) - (2\eta_n^2 - \lambda^2) f_n^{''}(z) + 2\lambda \eta_n^2 f_n^{'}(z) + (\frac{\eta_n^2 \lambda^2 \nu}{1-\nu} + \eta_n^4) f_n(z) \right] \sin \eta_n x \\ + \left[A_n^{''''}(z) - 2\lambda A_n^{'''}(z) + \lambda^2 A_n^{''}(z) \right] x + \left[B_n^{''''}(z) - 2\lambda B_n^{'''}(z) + \lambda^2 B_n^{''}(z) \right] = 0 \end{gathered} \tag{16}$$

As relation (16) must be valid for all values of x, thus:

$$f_n^{''''}(z) - 2\lambda f_n^{'''}(z) - (2\eta_n^2 - \lambda^2) f_n^{''}(z) + 2\lambda \eta_n^2 f_n^{'}(z) + (\frac{\eta_n^2 \lambda^2 \nu}{1-\nu} + \eta_n^4) f_n(z) = 0 \tag{17}$$

$$A_n^{''''}(z) - 2\lambda A_n^{'''}(z) + \lambda^2 A_n^{''}(z) = 0 \tag{18}$$

$$B_n''''(z) - 2\lambda B_n'''(z) + \lambda^2 B_n''(z) = 0 \tag{19}$$

At the two ends because of simply support condition we have:

$$w_n(0,z) = w_n(L,z) = 0 \tag{20}$$

$$\sigma_{xx,n}(0,z) = \sigma_{x,nx}(L,z) = 0 \tag{21}$$

where w is the displacement in z-direction.(also we define u as displacement in x-direction). By considering (11-a) and $w_n(0,z) = 0$ from (21), it can be concluded

$$B_n(z) = b_{1n} z + b_{2n} \tag{22}$$

By considering (11-a), $w_n(0,z) = L$ from (21) and (22), the result is

$$A_n(z) = a_{1n} z + a_{2n} \tag{23}$$

By considering (12-b) and this fact that there is no shear at surfaces z=h and z=0, it can be resulted

$$A_n'(0) = 0\,,\; A_n'(h) = 0\,,\; f_n'(0) = 0 \text{ and } f_n'(h) = 0 \tag{24}$$

from (24) and (26) it can be concluded that $a_1 = 0$ or

$$A_n(z) = a_{2n} \tag{25}$$

Linear terms in stress function produce no stress. Thus by considering (25), (22) and (9), we drop out linear terms and rewrite stress function as

$$\phi(x,z) = \frac{\bar{f}_n(z)}{\eta_n^2} \sin \eta_n x \tag{26}$$

relation (15) is a 4th order ordinary diffrential equation and its solution obtained from combination of four independent term. For solving this equation we assume the solution as $\bar{f}_n(z) = e^{\alpha_n z}$ and insert it in the equation. Then we obtain a characteristic equation as:

$$\alpha_n^4 - 2\lambda\alpha_n^3 - 2(\eta_n^2 - \frac{\lambda^2}{2})\alpha_n^2 + 2\lambda\eta_n^2\alpha_n + \left(1 - \nu + \frac{\nu\lambda_n^2}{\eta_n^2}\right)\frac{\eta_n^4}{1-\nu} = 0 \tag{27}$$

If (27) has four distinict real root, then we have

$$\bar{f}_n = \rho_{n1} e^{\alpha_{n1} z} + \rho_{n2} e^{\alpha_{2n} z} + \rho_{n3} e^{\alpha_{3n} z} + \rho_{4n} e^{\alpha_{4n} z} \tag{28}$$

If there is multiple root, we have terms like $x\rho_{n1} e^{\alpha_{n1} z}$ and if there is imaginary root, we have terms like $\cos\beta_{n1} x\ e^{\alpha_{n1} x}$ in (28).

Four unknown constants $\rho_{n1}, \rho_{n2}, \rho_{n3}, \rho_{n4}$ are determined from four conditions in (10) and (24). By obtaining these constants, the stress field is then known using (11).

For obtaining displacement field we first calculate strain components. By assumption state of plain strain, according to (11), (13) and (29) we have:

$$\varepsilon_{xx} = \sum_n \frac{1}{E_0}\left[\frac{1-\nu^2}{\eta_n^2}\sum_i \rho_{ni}\alpha_{ni}^2 e^{(\alpha_{ni}-\lambda)z} + \nu(\nu+1)\sum_i \rho_{ni} e^{(\alpha_{ni}-\lambda)z}\right]\sin\eta_n x \tag{29}$$

$$\varepsilon_{zz} = \sum_n \frac{1}{E_0}\left[\frac{-\nu(\nu+1)}{\eta_n^2}\sum_i \rho_{ni}\alpha_{ni}^2 e^{(\alpha_{ni}-\lambda)z} - (1-\nu^2)\sum_i \rho_{ni} e^{(\alpha_{ni}-\lambda)z}\right]\sin\eta_n x \tag{30}$$

$$\gamma_{zx} = 2\varepsilon_{zx} = \sum_n \frac{-2(1+\nu)}{E_0}\left[\sum_i \rho_{ni}\alpha_{ni} e^{(\alpha_{ni}-\lambda)z}\right]\cos\eta_n x \tag{31}$$

But $\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$, thus integration of (29) results

$$u=\sum_{n}\frac{-1}{E_0\,\eta_n}\left[\frac{1-\nu^2}{\eta_n^2}\sum_i\rho_{ni}\alpha_{ni}^2e^{(\alpha_{ni}-\lambda)z}+\nu(\nu+1)\sum_i\rho_{ni}e^{(\alpha_{ni}-\lambda)z}\cos\eta_n x+L_n(z)\right] \qquad (32)$$

By considering $\varepsilon_{zz}=\dfrac{\partial w}{\partial z}$ and (30) we obtain

$$w=\frac{1}{E_0}\left[\frac{-\nu(\nu+1)}{\eta^2}\sum_i\frac{\rho_i\alpha_i^{\;2}}{\alpha_i-\lambda}e^{(\alpha_i-\lambda)z}-(1-\nu^2)\sum_i\frac{\rho_i}{\alpha_i-\lambda_{\,i}}\;e^{(\alpha_i-\lambda)z}\right]\sin\eta x+m(x) \qquad (35)$$

but $\gamma_{zx}=(\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x})$ and using (33), (34) and (35) results:

$L^{'}(z)+m^{'}(x)=0$

and thus:

$L(z)=az+b$ (36)

$m(x)=-ax+d$ (37)

now from (37), first in (21) and (35) we conclude:

$d=0$ (38)

and from (37), second in (21) , (35), 38 and $\sin\eta L=0$ we conclude:

$a=0$ (39)

therefor:

$$u\;=\frac{-1}{E_0\eta}\left[\frac{1-\nu^2}{\eta^2}\sum_i\rho_i\alpha_i^{\;2}e^{(\alpha_i-\lambda)z}+\nu(\nu+1)\sum_i\rho_ie^{(\alpha_i-\lambda)z}\right]\cos\eta x+b \qquad (40)$$

$$w=\frac{1}{E_0}\left[\frac{-\nu(\nu+1)}{\eta^2}\sum_i\frac{\rho_i\alpha_i^{\;2}}{\alpha_i-\lambda}e^{(\alpha_i-\lambda)z}-(1-\nu^2)\sum_i\frac{\rho_i}{\alpha_i-\lambda_{\,i}}\;e^{(\alpha_i-\lambda)z}\right]\sin\eta x \qquad (41)$$

now it is necessary only to consider a boundary condition for u, as a example $u(0,z)=0$ to determine constant b in (40) and then displacement field is known. Results in [12] are valid if $u(\dfrac{L}{2},z)=0$. If this condition considered and n assumed to be odd number then, $\cos\dfrac{\eta L}{2}=0$ and b must be zero. In this state the assumed displacement in [12] is valid.

Conclusions

An elasticity solution is obtained for simply supported functionally gradient beams subjected to sinusoidal transverse loading. The Poisson ratio is assumed to be a constant, and the Young’s modulus is assumed to vary in an exponential fashion through the thickness. Every loding that its intensity are zero at two ends, can written in a sinusoidal foureir series, and thus can be solved in this manner.

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