Bending Design of RC Circular Cross Sections according to ENV 1992 Standard for Strains $\varepsilon_{c2} \geq 1.35$ ‰

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Summary

The paper discusses bending analysis of a reinforcement concrete circular cross section implementing the bilinear design stress-strain relationship for the concrete as given by ENV 1992 standard. It covers the case when the maximal strain in concrete ε_{c2} that appears at the more strongly compressed edge of the compressive zone exceeds the value of 1.35 ‰ and the stresses are partially described by a linear function and partially by a constant value. For this case the paper presents the development of the analytical expressions for coefficients α_v and k_a. The procedure for determination of required cross section A_{s1}, based on equations resulting from equilibrium conditions, is also briefly described.

Introduction

The analysis procedures for circular cross sections deviate essentially from the analysis of other types of cross sections due to a nonlinear change in the width of the cross section.

The paper implements the bilinear design stress-strain relationship for the concrete as given by the ENV 1992 standard [1] (or shortly EC2). Two possibilities result from the value of the maximal strain in concrete ε_{c2} that appears in the more strongly compressed edge of the compressive zone. The first case, in which the absolute maximal strain in the concrete is smaller or equal to 1.35 ‰, and the stresses in the concrete have only a linear distribution, is already covered in references [2]. If the compressive stresses in the top compressed edge exceed the value of 1.35 ‰, the stress-strain relation is linear for the strains below 1.35 ‰, and stresses have a constant value for strains over 1.35 ‰. For this case, the paper presents the development of the analytical expressions for coefficients α_v (the average value of concrete stresses σ_m in the bending compression zone, related to the design value of concrete strength f_{cd}) and k_a (the related distance of the concrete compression force from the stronger compressed edge of the compressive cross section). These analytical expressions are necessary for the computation of the required cross section of the reinforcements A_{s1} and A_{s2} (if the compression reinforcement is required).

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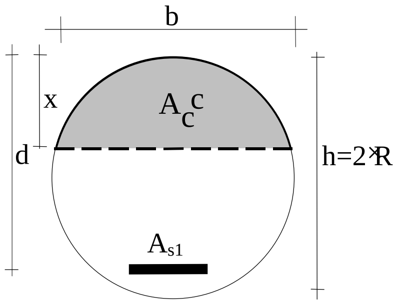
Symbols and notations

The notations considering the dimensions of a circular cross sections used in the paper are in accordance with in the EC2 and are presented in Figure 1:



h	overall depth of cross-section,
b	variable width of cross-section,
x	neutral axis depth,
d	effective depth of cross-section,
A_{s1}	area of the reinforcement,
A_c^c	area of the concrete under compression.

$$A_c^c = \frac{-x\cdot 2\cdot R^2 - 3\cdot R\cdot x + x^2}{2\cdot\sqrt{(2\cdot R - x)\cdot x}} + R^2\cdot \mathrm{Atan}\left(\sqrt{\frac{x}{2\cdot R - x}}\right)$$

Figure 1. A circular cross section.

Basic assumptions for flexural analysis process

EC2 allows the utilization of three different stress-strain design diagrams for concrete: simple truncated rectangle, bilinear diagram and the parabola-rectangular diagram. Although these diagrams differ in the mathematical description of stresses, the maximal compressive strain in concrete for bending analysis in all three cases is limited to the value 3.5 ‰. In the bilinear diagram, that is used trough this paper, the margin between the linear stress distribution and the constant value of stress lies at the strain 1.35 ‰ in compression, regardless of the compressive strength of the concrete. The linear distribution of stresses in the concrete is expressed as a function of the strain ε_c (just for the convenience the absolute value of the strain value ε_c, directly in ‰, is used), as:

$$\sigma_c = \frac{\varepsilon_c}{1.35}\cdot f_{cd} \qquad \varepsilon_c \le 1.35\,‰ \tag{1}$$

Further, the stresses in the concrete for strains higher than 1.35 ‰ are equal to f_{cd}, which represents the design compressive strength of concrete.

The uniaxial bending analysis of circular cross sections is based on the assumption of Bernoulli's-hypothesis (all adjacent plane cross sections remain plane during loading). This assumption assures linear distribution of strains over the cross section. All strains in the cross section, that are governed by the strains ε_{c2} and ε_{s1}, i.e. strain in the more strongly compressed edge of the compressive zone and the strain in reinforcement in the tensile zone, respectively, can be found by linear interpolation between the margin values ε_{c2} and ε_{s1}. As a consequence of strains, stresses develop in the deformed region. In the concrete below the neutral axis, which is considered to be cracked, stresses cannot develop and so this area does not contribute to the fulfillment of static equilibrium. Therefore the cracked concrete is replaced by the steel reinforcement.

Derivation of coefficients α_V and k_A for circular cross sections

For the case under investigation, i.e. $\varepsilon_{c2} \geq 1.35$ ‰, the stresses in the concrete are described partly by linear distribution and partly by a constant value. The margin between the two mathematical descriptions lies at $\varepsilon_c = 1.35$ ‰ and the distance from top fibre and the fibre with strain 1.35 ‰ is denoted with y_m. The strains are expressed as a function of the distance from the neutral axis y and the strain value ε_{c2}:

$$\varepsilon_c(y) = \left(1 - \frac{y}{x}\right) \cdot \varepsilon_{c2} \tag{2}$$

The "stress body" of the compresses zone or the resultant of stresses in concrete is thus obtained implementing Eqs. (1) and (2) by an integral as:

$$F_c = \int_{A_c^c} \sigma_c \cdot dA = \int_{y=0}^{x} \int_{z=-\frac{b(y)}{2}}^{\frac{b(y)}{2}} \sigma_c \cdot dz \cdot dy = \int_{y=0}^{y_m} f_{cd} \cdot b(y) \cdot dy + \int_{y=y_m}^{x} \sigma_c \cdot b(y) \cdot dy \tag{3}$$

The fullness factor is further obtained by dividing the obtained result with the cross section area of concrete in compression A_c^c and concrete strength f_{cd}. The final result is:

$$\alpha_v = \frac{\alpha_n}{\alpha_d} \tag{4}$$

with (coefficients a and p are just abbreviations):

$$a_1 = -27 + 20 \cdot \varepsilon_{c2} \qquad a_2 = -20 + \frac{27}{\varepsilon_{c2}} \qquad a_3 = 2 \cdot R - x$$

$$a_4 = \sqrt{\frac{a_1 \cdot x \cdot (40 \cdot \varepsilon_{c2} \cdot R - a_1 \cdot x)}{\varepsilon_{c2}^2}}$$

$$p_1 = -48000 \cdot \varepsilon_{c2}^2 \cdot a_1 \cdot R^2 \cdot (R - x) \cdot a_3 \cdot x \cdot \sqrt{40 \cdot R + a_2 \cdot x} \cdot \mathrm{Atan}\left(\sqrt{\frac{x}{a_3}}\right)$$

$$p_2 = 2400 \cdot \varepsilon_{c2}^2 \cdot R^2 \cdot (a_1 \cdot x - 20 \cdot \varepsilon_{c2} \cdot R) \cdot \sqrt{-a_2 \cdot x} \cdot a_4$$
$$\cdot \mathrm{Atan}\left(\frac{\varepsilon_{c2} \cdot \sqrt{-a_2 \cdot x} \cdot \sqrt{40 \cdot R + a_2 \cdot x}}{-27 \cdot x - 20 \cdot \varepsilon_{c2} \cdot a_3}\right)$$

$$p_3 = 400 \cdot \varepsilon_{c2}^2 \cdot \left(3 \cdot R^2 - 2 \cdot R \cdot x + x^2\right) \cdot \left(20 \cdot \sqrt{a3 \cdot x} - a_4\right) - 1080 \cdot \varepsilon_{c2} \cdot (R - x) \cdot x \cdot a_4$$
$$- 729 \cdot x^2 \cdot \sqrt{\frac{a_1 \cdot x \cdot (40 \cdot \varepsilon_{c2} \cdot R - a_1 \cdot x)}{\varepsilon_{c2}^2}}$$

$$\alpha_n = 32400 \cdot \varepsilon_{c2} \cdot a_1 \cdot a_3 \cdot x \cdot \sqrt{40 \cdot R + a_2 \cdot x} \cdot \left(2 \cdot R^2 \cdot \sqrt{a_3 \cdot x} \cdot \mathrm{Atan}\left(\sqrt{\frac{x}{a_3}}\right) - x \cdot \left(2 \cdot R^2 - 3 \cdot R \cdot x + x^2\right)\right) \tag{5}$$

$$\alpha_d = \sqrt{a_3 \cdot x} \cdot \left(p_1 + a_3 \cdot \left(a_1 \cdot x \cdot \sqrt{40 \cdot R + a_2 \cdot x} \cdot p_3 + p_2\right)\right) \tag{6}$$

Similarly, the bending moment of stresses over the more strongly compressed edge of the compressive zone is obtained by the integral:

$$\int_{y=0}^{y_m} f_{cd} \cdot y \cdot b(y) \cdot dy + \int_{y=y_m}^{x} \sigma_c \cdot y \cdot b(y) \cdot dy \tag{7}$$

and coefficient k_a is finally obtained by dividing the obtained result by the resultant of compressive stresses in concrete, which results in:

$$k_a = \frac{\alpha_n}{\alpha_d} \tag{8}$$

with:

$$p_5 = -480000 \cdot {\varepsilon_{c2}}^2 \cdot a_1 \cdot R^3 \cdot (5 \cdot R - 4 \cdot x) \cdot a_3 \cdot x \cdot p_4 \cdot \mathrm{Atan}\left(\sqrt{\frac{x}{a_3}}\right)$$
$$p_6 = 14580 \cdot \varepsilon_{c2} \cdot (R - 3 \cdot x) \cdot x^2 \cdot a_4 + 19683 \cdot x^3 \cdot a_4 + 5400 \cdot {\varepsilon_{c2}}^2 \cdot x \cdot a_4 \cdot \left(-7 \cdot R^2 - 4 \cdot R \cdot x + 6 \cdot x^2\right)$$
$$p_7 = 4000 \cdot {\varepsilon_{c2}}^3 \cdot \left(15 \cdot R^3 - 7 \cdot R^2 \cdot x - 2 \cdot R \cdot x^2 + 2 \cdot x^3\right) \cdot \left(20 \cdot \sqrt{a_3 \cdot x} - a_4\right)$$
$$p_8 = -1080 \cdot \varepsilon_{c2} \cdot (R - x) \cdot x \cdot a_4 - 729 \cdot x^2 \cdot a_4 + 400 \cdot {\varepsilon_{c2}}^2 \cdot \left(3 \cdot R^2 - 2 \cdot R \cdot x + x^2\right) \cdot \left(20 \cdot \sqrt{a_3 \cdot x} - a_4\right)$$

$$\alpha_n = p_5 + a_3 \cdot \left(a_1 \cdot x \cdot p_4 \cdot (p_6 + p_7) + 96000 \cdot {\varepsilon_{c2}}^3 \cdot R^3 \cdot \sqrt{-a_2 \cdot x} \cdot a_4 \cdot (a_1 \cdot x - 25 \cdot \varepsilon_{c2} \cdot R) \cdot \mathrm{Atan}\left(\frac{\varepsilon_{c2} \cdot \sqrt{-a_2 \cdot x} \cdot p_4}{-27 \cdot x - 20 \cdot \varepsilon_{c2} \cdot a3}\right)\right) \tag{9}$$

$$\alpha_d = 40 \cdot \varepsilon_{c2} \cdot x \cdot \left(p_1 + a_3 \cdot \left(a_1 \cdot x \cdot p_4 \cdot p_8 + p_2\right)\right) \tag{10}$$

Procedure for the Determination of Required Reinforcement A_{S1}

Two equilibrium conditions must be satisfied when the cross section is subjected to the design value of bending moment M_{Sd}. The equilibrium of axial forces can be written as:

$$F_c - F_{s1} = \alpha_v \cdot \alpha \cdot f_{cd} \cdot A_c{}^c - A_{s1} \cdot \sigma_{s1} = 0 \tag{11}$$

Also the equilibrium of bending moments must be preserved for an arbitrary point. The most frequently used points are the centre of reinforcements and the centre of the force F_c:

$$\Sigma M_s = 0 \tag{12}$$

$$F_c \cdot z - M_{Sd} = A_c{}^c \cdot \alpha_v \cdot \alpha \cdot f_{cd} \cdot z - M_{Sd} = A_c{}^c \cdot \alpha_v \cdot \alpha \cdot f_{cd} \cdot \zeta \cdot d - M_{Sd} = 0 \tag{13}$$

$$\Sigma M_c = 0 \tag{14}$$

$$F_s \cdot z - M_{Sd} = A_{s1} \cdot \sigma_{s1} \cdot z - M_{Sd} = A_{s1} \cdot \sigma_{s1} \cdot \zeta \cdot d - M_{Sd} = 0 \tag{15}$$

where z is lever arm of internal forces and ζ is the related distance of the lever arm of internal forces (dimensionless). It is expressed as the ratio of the lever arm of internal forces z over effective depth of cross-section d. The equations (13) and (15) actually represent identical equation, just written in two different forms. The problem thus reduces to the determination of several unknowns that must satisfy two equilibrium equations. If the dimensions of the cross section are given (which is the most frequent case) and the position of the reinforcement d is selected ($d<h$ and according to the minimum concrete cover and other requirements), the only unknowns left are ε_{s1}, ε_{c2}, and A_{s1}.

To solve the problem while only two equations are available one of the three left unknowns is chosen within allowed limits. As the reinforcement cross section A_{s1} is directly related to stresses in the steel σ_1 that further depend on the value of strains in the steel ε_{s1}, this unknown cannot be selected and must be determined as the last one. In the engineering analysis the value of strains in concrete ε_{c2} is usually selected for economic reasons and the strains in steel ε_{s1} are afterwards computed in an iterative procedure from equilibrium conditions. Starting with the initial value of ε_{s1}, the position of the neutral axis x can be computed as:

$$x = \frac{\varepsilon_{c2}}{\varepsilon_{s1} + \varepsilon_{c2}} \cdot d \tag{16}$$

According to the position of the neutral axis coefficients α_v and k_a are evaluated implementing expression (4) and (8), respectively. Force in the concrete is now:

$$F_c = \alpha_v \cdot \alpha \cdot f_{cd} \cdot A_c^{\ c} \quad (17)$$

Further, the lever arm of internal forces z is computed as:

$$z = \zeta \cdot d = (1 - k_a \cdot \xi) \cdot d = \left(1 - k_a \cdot \frac{\varepsilon_{c2}}{\varepsilon_{s1} + \varepsilon_{c2}}\right) \cdot d \quad (18)$$

If the equilibrium of bending moments (Eq. (13) with α=0.80) is not satisfied, new value of ε_{s1} is selected and the procedure is repeated. However, ff the equilibrium of axial forces is satisfied, the required cross section of reinforcement A_{s1} is finally computed by dividing the force in concrete F_c by $\sigma_{s1,}$ where the actual stress in reinforcement σ_{s1} is computed depending on the value of strains in steel ε_{s1}.

Conclusions

The uniaxial bending analysis of circular cross sections was considered using Bernoulli's-hypothesis and the standard EC2. The bilinear stress-strain diagram for concrete was adopted and the case where the strains at the more compressed edge of the compressive zone exceeded the value of 1.35 ‰ was considered. For this case the development of analytical expressions for the coefficients α_v and k_a was presented. The derived coefficients α_v and k_a, implemented with the iterative procedure described, allow the reliable analysis of the circular cross sections as all the parameters required are accurately determined from equilibrium equations.

All presented expressions are given in analytical form and therefore they represent a very suitable form for computer programs because it is possible to use the described iterative scheme to determine the required reinforcement cross section A_{s1}.

Reference

1 ENV 1992-1-1: Eurocode 2: *Design of concrete structures. Part 1: General rules and rules for buildings*, 1991.

2 Skrinar, M. (2003): ``Bending design of reinforced concrete circular cross sections after EC2 standard for strain in concrete smaller than 1.35 ‰, *Proceedings in applied mathematics and mechanics*, Vol. 3, iss. 1, pp. 314-315.