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Exact Static Stiffness Matrix for a Deep Beam Embedded in an Elastic Medium

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Summary

The fourth order differential equation governing the static displacement of a beam embedded in an elastic medium is developed from first principles. Allowance is made for the effects of shear deformation and the distributed lateral and rotational stiffness of the supporting medium. Each effect is defined by a unique, non-dimensional parameter whose value can be set to zero if the particular effect is not to be considered. This enables any combination of effects to be accounted for. The governing differential equation is then solved and the solution stated in the form of an exact static stiffness matrix. The resulting equations find considerable application when considering problems involving soil-structure interaction.

Introduction

The allowance for shear deformation is becoming progressively more important as structural elements are more regularly fabricated from composite materials that are very strong in their primary load carrying directions, but are often weak in shear [1, 2]. This relative weakness manifests itself in a low shear modulus to Young's modulus ratio, which is typically three to four times less than the equivalent metallic member and can be as much as ten times less [3]. In addition, there is now a more regular need to analyse structures whose supports can 'give' under applied load in a more rigorous way than has been customary hitherto. Such structures may range from railway sleepers to strip foundations, to buried pipelines and in a more complex environment, the coupled movement of piles and pile groups.

The present paper formulates from first principles the differential equation that governs the lateral displacement of a beam member that is embedded in an elastic medium. It allows for the coupled effects of shear deformation and the distributed lateral and rotational support stiffnesses provided by the medium. The theory is developed in terms of non-dimensional parameters that uniquely describe the effect of shear, distributed lateral support stiffness and distributed rotational support stiffness, in such a way that any effect can be included by assigning the corresponding parameter its correct value, or ignored if the relevant

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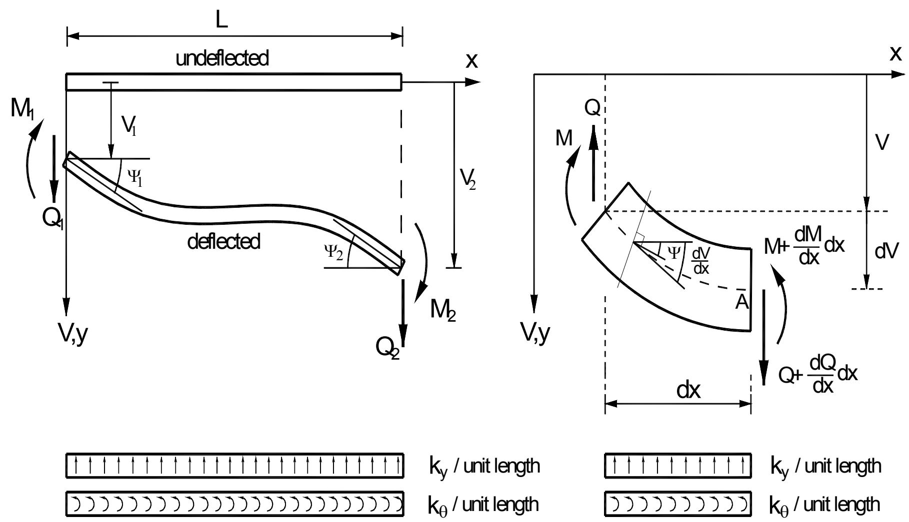
parameter is set to zero. The solution of the governing differential equation leads to an exact stiffness matrix formulation that can be used in the normal way within the structure of the finite element technique.



(a) (b)

Figure 1. (a) Positive directions of the nodal forces and displacements in member co-ordinates; (b) Positive directions of the forces and displacements acting on an elemental length of the member in local co-ordinates.

Theory

Figure 1(b) shows the forces and displacements associated with a typical elemental length, *dx*, of a beam of length *L* that is subject to lateral and rotational distributed support stiffnesses. Resolving vertically and taking moments about A gives, respectively,

$$dQ / dx = k_y V \quad \text{and} \quad Q = dM / dx + k_\theta \Psi \qquad (1, 2)$$

while the bending moment, bending slope and shear slope are given by [4]

$$M = -EI\, d\Psi / dx, \quad \Psi = dV / dx - \Gamma \quad \text{and} \quad \Gamma = Q / \phi \qquad (3, 4, 5)$$

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where $\phi = k'AG$, $k' =$ section shape factor [5], A = cross-sectional area, G = modulus of rigidity, E = Young's modulus, I = second moment of area of the member cross-section and Q, M, V, Ψ and Γ are the shear force, bending moment, lateral displacement, bending slope and shear slope, respectively, at a typical distance x from the left hand end of the member. k_y and k_θ are the lateral and rotational stiffnesses/unit length of the encapsulating medium.

Eliminating either V, Ψ or Γ from Eqs. (1) – (5) and introducing the non-dimensional parameter $\xi = x / L,$ the required differential equation that governs the displacement of the beam may be written as

$$\left[D^4 - (s^2 K_y + K_\theta) D^2 + K_y (1 + s^2 K_\theta)\right] \Delta = 0 \tag{6}$$

where $D = \dfrac{d}{d\xi}$, $\xi = x / L$, $L =$ member length, Δ = V, Ψ or Γ, the shear parameter $s^2 = EI / \phi L^2$, $K_y = k_y L^4 / EI$ and $K_\theta = k_\theta L^2 / EI$ (7)

This non-dimensional formulation is particularly convenient, since the effects of shear deformation, lateral distributed support stiffness and rotational distributed support stiffness, defined uniquely by the parameters s^2, K_y and K_θ, respectively, are included when the relevant parameter takes its natural value and omitted when the relevant parameter is set to zero.

If now we assume that $\Delta = V$ we may write the general solution for Eq. (6) as

$$V = C_1 \cosh \alpha\xi + C_2 \sinh \alpha\xi + C_3 \cosh \beta\xi + C_4 \sinh \beta\xi \tag{8}$$

where

$$\alpha = \left[b + (b^2 - c)^{½}\right]^{½}, \qquad \beta = \left[b - (b^2 - c)^{½}\right]^{½},$$

$$2b = s^2 K_y + K_\theta \quad \text{and} \quad c = K_y (1 + s^2 K_\theta) \tag{9}$$

The form of Eq. (8) presumes that α and β are always real, although inspection of Eq. (9) shows that they can also be imaginary or complex, depending on the values of b and c. Thus when programming the stiffness matrix of Eq. (21) it will be necessary either to work in complex arithmetic or to use

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alternative equations to Eq. (8) when appropriate [6] if it is necessary always to work in real arithmetic.

Equations defining Ψ, Q and M can be determined in non-dimensional form from Eqs. (1) – (5) together with Eqs. (7) as

$$\Psi \left(1+s^2 K_\theta\right)L = \left[1+s^2\left(D^2 - s^2 K_y\right)\right]DV \tag{10}$$

$$Q \left(1+s^2 K_\theta\right)L^3 = -EI\left[D^2 - s^2 K_y - K_\theta\right]DV \tag{11}$$

$$ML^2 = -EI\left[D^2 - s^2 K_y\right]V \tag{12}$$

Substituting for V from Eq. (8), suitably differentiated, gives

$$\Psi = \gamma_\psi\left[C_1\alpha_1 \sinh\alpha\xi + C_2\alpha_1\cosh\alpha\xi + C_3\beta_1\sinh\beta\xi + C_4\beta_1\cosh\beta\xi\right] \tag{13}$$

$$Q = -\gamma_q\left[C_1\alpha_2\, sinh\,\alpha\xi + C_2\alpha_2\, cosh\,\alpha\xi + C_3\beta_2\, sinh\,\beta\xi + C_4\beta_2\, cosh\,\beta\xi\right] \tag{14}$$

$$M = -\gamma_m[C_1\alpha_3\, cosh\,\alpha\xi + C_2\alpha_3\, sinh\,\alpha\xi + C_3\beta_3\, cosh\,\beta\xi + C_4\,\beta_3\, sinh\,\beta\xi] \tag{15}$$

where

$$\begin{aligned}
\alpha_1 &= \alpha[1+s^2(\alpha^2 - s^2 K_y)], & \beta_1 &= \beta[1+s^2(\beta^2 - s^2 K_y)]\\
\alpha_2 &= \alpha[\alpha^2 - s^2 K_y - K_\theta], & \beta_2 &= \beta[\beta^2 - s^2 K_y - K_\theta]\\
\alpha_3 &= \alpha^2 - s^2 K_y, & \beta_3 &= \beta^2 - s^2 K_y
\end{aligned}$$

and

$$\gamma_\psi = 1/(1+s^2 K_\theta)L, \quad \gamma_q = EI/(1+s^2 K_\theta)L^3, \quad \gamma_m = EI/L^2 \tag{16}$$

The nodal forces and displacements can now be defined in the member co-ordinate system of Figure 1(a), as follows:

at node 1, the left hand end of the member,

$$\xi = 0: \quad V_1 = V,\ \Psi_1 = \Psi,\ Q_1 = -Q,\ M_1 = M$$

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at node 2, the right hand end of the member,

$$\xi = 1: \qquad V_2 = V,\ \ \Psi_2 = \Psi,\ \ Q_2 = Q,\ \ M_2 = -M \tag{17}$$

Thus the nodal displacements and forces can be determined from Eqs. (8), (13), (14) and (15) and can be written in matrix form as:

$$\begin{bmatrix} V_1 \\ \Psi_1 \\ V_2 \\ \Psi_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \gamma_\psi \alpha_1 & 0 & \gamma_\psi \beta_1 \\ \cosh\alpha & \sinh\alpha & \cosh\beta & \sinh\beta \\ \gamma_\psi \alpha_1 \sinh\alpha & \gamma_\psi \alpha_1 \cosh\alpha & \gamma_\psi \beta_1 \sinh\beta & \gamma_\psi \beta_1 \cosh\beta \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \tag{18}$$

or

$$\mathbf{d} = \mathbf{S}\,\mathbf{C} \tag{18a}$$

and

$$\begin{bmatrix} Q_1 \\ M_1 \\ Q_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} 0 & \gamma_q \alpha_2 & 0 & \gamma_q \beta_2 \\ -\gamma_m \alpha_3 & 0 & -\gamma_m \beta_3 & 0 \\ -\gamma_q \alpha_2 \sinh\alpha & -\gamma_q \alpha_2 \cosh\alpha & -\gamma_q \beta_2 \sinh\beta & -\gamma_q \beta_2 \cosh\beta \\ \gamma_m \alpha_3 \cosh\alpha & \gamma_m \alpha_3 \sinh\alpha & \gamma_m \beta_3 \cosh\beta & \gamma_m \beta_3 \sinh\beta \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \tag{19}$$

or

$$\mathbf{p} = \mathbf{S}^*\,\mathbf{C} \tag{19a}$$

Hence from (18a)

$$\mathbf{C} = \mathbf{S}^{-1}\,\mathbf{d} \tag{20}$$

Substituting Eq. (20) into Eq. (19a) yields the required stiffness relationship as

$$\mathbf{p} = \mathbf{k}\,\mathbf{d} \tag{21}$$

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where

$$\mathbf{k} = \mathbf{S}^{*}\,\mathbf{S}^{-1} \tag{22}$$

Conclusions

The Summary can be read as a statement of the conclusions.

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