A Multi-level Interpolation Method for Solving Elliptic Partial Differential Equations with Variable Coefficients

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Summary

A meshless method based on the idea of direct multi-elliptic interpolation is presented. The solution of the original problem is approximated by a special multi-elliptic interpolation involving also the original differential operator. The interpolation is performed on a quadtree/octtree cell system. This results in a fast, stable solution algorithm, which avoids also the use of large, dense and ill-conditioned matrices.

Introduction

Meshless (meshfree) methods have become quite popular in the last decade, since one of the most difficult tasks in solving partial differential equations is a proper mesh or grid generation, especially in 3D problems. Meshless methods do not require any mesh structure, only an unstructured set of points in the computational domain. The price of this advantage is that the computational schemes are often more complicated and/or more expensive from numerical point of view.

Perhaps the most popular truly meshfree method is the collocation method with radial basis functions (RBFs). This approach is based on a scattered data interpolation, when a function u is approximated in the following form:

$$u(x) \sim \sum_{j=1}^{N} \alpha_j \Phi(x - x_j) \tag{1}$$

(i.e. as a linear combination of shifted RBFs), supplied with the interpolation conditions at the interpolation points $x_1, x_2, ..., x_N$:

$$u(x_k) := u_k \quad (k = 1,2,..., N) \tag{2}$$

Here the interpolation points $x_1, x_2, ..., x_N$ are scattered in a 2D or 3D domain Ω (depending on the problem considered). The radial basis function Φ can be chosen in various ways. The most frequently appearing RBFs are the mul-

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tiquadrics ($\Phi(r) = \sqrt{r^2 + c^2}$ with a scaling parameter c), the thin plate splines, or, more generally, the polyharmonic splines ($\Phi(r) = r^{2m} \log r$). Another direction of generalization is the use of fundamental solutions of certain multi-elliptic operators (e.g. $\Phi(r) = crK_1(cr)$, the fundamental solution of the bi-Helmholtz operator), see [6,7] for details.

The simplest way to construct meshfree methods based on scattered data interpolation is to substitute the interpolation function into the original partial differential equation and also into the boundary conditions at the interpolation points which are located along the boundary (*Kansa's method* [8]). This results in a generally nonsymmetric system of equations for the unknown coefficients of the interpolation. Using a Hermite-based approach, this system can be symmetrized [4], which has computational advantages. If the applied RBF itself (nearly) satisfies the original partial differential equation to be solved, it is sufficient to use boundary interpolation points only, and we arrive at the *boundary meshless methods*. A variant of this approach, the *method of fundamental solution* is based on the fundamental solution of the original partial differential equation. In usual second-order problems, the fundamental solution exhibits singularity at the origin, which causes numerical difficulties. The problem of the singularity can be avoided by introducing fictitious points outside the domain or by regularization. In the case of the Laplace-Poisson equation, the fundamental solution of the operator $\Delta(\Delta - c^2 I)$ can be used ($\Phi(r) = K_0(cr) + \log(cr)$, the singularities cancel out) instead of the fundamental solution of the Laplacian, with a carefully chosen scaling parameter c, see [7]. It is also possible to completely avoid the problem of singular functions by using nonsingular general solutions (*boundary knot method* [3]).

However, almost all of the above approaches suffer from a common computational disadvantage. Namely, the resulting linear system of equations is large, full and severely ill-conditioned in general. Consequently, their application to really large scale problems is difficult. The computational cost can – and should – be reduced by sophisticated techniques such as domain decomposition; fast multipole evaluation [1] or the use of compactly supported RBFs and/or multilevel techniques [2].

Our approach is based on the direct multi-elliptic method [6], which provides an RBF-like interpolation function using the fundamental solution of various (at least fourth order) multi-elliptic operators as a radial basis function. It should be pointed out, however, that this approach circumvents the use of large, full and ill-conditioned matrices by directly solving a higher order auxiliary multi-elliptic problem supplied with the interpolation conditions as special boundary condi-

tions. The domain of this auxiliary problem can be defined in a practically arbitrary way (completely independently of the original problem), which makes it possible to apply robust, quadtree/octtree-based multi-level solution methods. This results in fast and stable algorithms, even if the distance of some interpolation points is small. The number of arithmetic operations is typically $O(N \log N)$ where N is the number of interpolation points.

In this paper, we generalize the idea of the direct multi-elliptic interpolation for more general elliptic problems with variable coefficients.

Meshfree Methods Based on the Direct Multi-Elliptic Interpolation

The Direct Multi-Elliptic Interpolation Method defines an interpolation function as a solution of an at least fourth order elliptic equation:

$$L_0 u = 0 \quad \text{in } \Omega_0 \setminus \{x_1, x_2, ..., x_N\} \tag{3}$$

(in the sense of distributions), supplied with the interpolation conditions (2). There are many possibilities for choosing L_0 e.g. the biharmonic operator ($L_0 := \Delta\Delta$); the bi-Helmholtz operator ($L_0 := (\Delta - c^2 I)^2$); the mixed Laplace-Helmholtz-operator ($L_0 := \Delta(\Delta - c^2 I)$), or even higher order operators. The domain Ω_0 can be defined to be as simple as possible. Along the boundary of Ω_0, any regular (e.g. Dirichlet or Neumann) boundary condition can be prescribed: the choice of the boundary condition plays only a minor role in the interpolation. The solvability of (3) can be analyzed by variational theory. In contrast to the usual second-order partial differential operators, the appearing fundamental solutions are continuous at the origin and the boundary conditions taken at discrete points do *not* destroy the well-posedness of the problem [6,7].

To economically solve the problem (2)-(3), multi-level quadtree-based techniques can be applied (octtree-based in 3D). The quadtree (QT-) algorithm defines a non-uniform cell system with local refinements at the interpolation points. The whole procedure can be fully automated. Finite volume schemes can easily be introduced in QT-cell systems in a natural multi-level way. For details, see [5].

This interpolation technique is suitable to construct meshfree methods. Suppose, for simplicity, that the original problem is the Dirichlet problem of the Poisson equation in a domain Ω:

$$\Delta u = f \quad \text{in } \Omega, \quad u = u_0 \quad \text{along } \partial\Omega \tag{4}$$

Let the domain Ω and the boundary $\partial\Omega$ be discretized by the points $x_1, x_2, ..., x_N \in \Omega$ and $x_{N+1}, x_{N+2}, ..., x_{N+M} \in \partial\Omega$, respectively. Now Kansa's method leads to a tetraharmonic-type interpolation [7]. A much simpler approach is the utilization of the idea of the particular solutions. First, the function *f* is approximated by a multi-elliptic e.g. a biharmonic interpolation in a larger domain $\Omega_0 \supset \Omega$. Next, *using the same QT-cell system*, a particular solution *U* of the Poisson equation (4) is constructed in Ω_0. Finally, the solution of (4) is expressed as $u = U + w$, where *w* solves the Laplace equation; the function *w* is approximated in a meshfree way by solving the Laplace-Helmholtz equation

$$\Delta(\Delta - c^2 I)w = 0 \quad \text{in } \Omega_0 \setminus \{x_{N+1}, x_{N+2}, ..., x_{N+M}\}, \tag{4}$$

supplied with the modified boundary (interpolation) conditions:

$$w(x_k) = u_0(x_k) - U(x_k) \qquad (k = N+1, N+2, ..., N+M) \tag{5}$$

The crucial point is the proper choice of the scaling parameter *c*. If *c* is too large, then singularities are generated at the boundary interpolation points. If *c* is too small, then the function *w* fails to (nearly) satisfy the Laplace equation in the middle of the domain. Both cases result is poor approximation.

Application to Equations with Variable Coefficients

Now consider the more general elliptic problem:

$$Lu := \operatorname{div} \sigma \operatorname{grad} u = f \quad \text{in } \Omega, \quad u = u_0 \quad \text{along } \partial\Omega \tag{6}$$

with a given, positive function σ. Let the domain Ω and its boundary be discretized again by the unstructured set of points $x_1, x_2, ..., x_N \in \Omega$ and $x_{N+1}, x_{N+2}, \ldots, x_{N+M} \in \partial\Omega$, respectively.

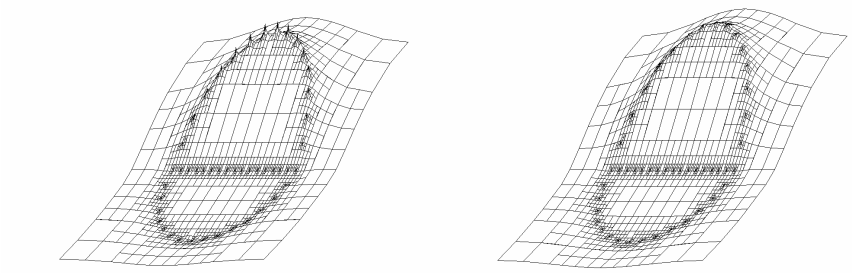
The fundamental solutions of the operator *L* are generally not radial functions (i.e. not circularly symmetric), and it is hopeless to compute them in an economic way. Thus, e.g. the method of fundamental solution or the BEM cannot be applied in their original form. However, the idea of the multi-elliptic interpolation still works. Let us apply separate multi-elliptic e.g. biharmonic interpolations to the functions *f* and σ in a larger domain $\Omega_0 \supset \Omega$. If the number of boundary points (*M*) is extremely large (their characteristic distance is below the

size of the finest QT-cells), then, of course, it is possible to directly solve the problem (6) in the same QT-cell system: for doing so, it is necessary to modify the finite volume schemes on the QT-cell system, see [5]. Otherwise, a particular solution U of (6) is constructed in the domain Ω_0 (using again the existing QT-cell system), and u is expressed as $u = U + w$, where w satisfies $Lw = 0$. To approximate this homogeneous solution in Ω, consider the multi-elliptic interpolation equation which is similar to (4):

$$L(L - c^2 I)w = 0 \quad \text{in } \Omega_0 \setminus \{x_{N+1}, x_{N+2}, ..., x_{N+M}\} \tag{7}$$

with the modified interpolation conditions (5). Note that Eq. (7) can be split into a pair of second-order equations $(L - c^2 I)w = v$, $Lv = 0$ .This makes the implementation simpler, since second-order operators have to be discretized only.

A numerical example: The features of the multi-elliptic method are illustrated through the following simple example. Let Ω be a circle and suppose that $\sigma \equiv 4$ in the lower half-circle and $\sigma \equiv 1$ in the upper half-circle, which results in a jump in the derivatives of the solution of (6) along the diameter: $u(x, y) = 4y$ ($y \geq 0$) and $u(x, y) = y$ ($y < 0$). The Dirichlet boundary condition was consistent with this solution. The boundary was discretized by 32 points and 15 additional points were located on the diameter where σ jumps. Figure 1 shows the approximate solutions of (7) on a QT-cell system (supplied with homogeneous Neumann boundary conditions). In case (*a*), logarithmic-type singularities can be observed at the interpolation points since the constant c was too high, and the relative error (measured in the discrete L_2-norm) was 8.58%. Setting c to 100, the singularities were removed and the relative error was reduced to 1.01% (case (*b*)).



(*a*) c = 500 (*b*) c = 100

Figure 1. Approximate solutions of the test problem on QT-cell systems

Remark: Another possible approach is to reconstruct first the boundary of the domain Ω (without any mesh structure) by solving the auxiliary Dirichlet problem $\Delta z = 0, \quad z|_{\partial\Omega_0} = 0$ supplied with $z_{N+1} = ... = z_{N+M} = 1$ as an interpolation condition. The QT-cells on which the computed values of z is greater than $1-\varepsilon$ with a predefined tolerance ε, are accepted as "inner" cells. Along the reconstructed boundary, interpolated boundary conditions are prescribed (applying a multi-elliptic interpolation to the boundary conditions), and now (6) can directly be solved on the reconstructed domain, using the same QT-cell system.

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