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Finite Element Analysis of Elastica

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Summary

Based on the newly derived variational principle of elastica exclusively expressed in terms of the rotation of beam, the finite element method is formulated for the analysis of plane deformation of elastica. Numerical results are compared with the analytical and numerical solutions, showing the effectiveness of the proposed finite element analysis of elastica.

Introduction

Finite element analysis of elastica has not appeared in the literature since the variational principle of elastica had not been clarified due to the fact that the inextensibility of the beam yield no axial strain energy. Recently, the variational principle of elastica exclusively expressed in terms of the rotation of beam was developed by Kondo [1].

Based on the newly derived variational principle, the finite element method is formulated in order to analyze the plane deformation of straight elastica. Numerical results show the effectiveness of the proposed finite element method for plane beam exhibiting the finite rotations as well as the finite displacements.

Principle of Virtual Work for Elastica

We consider a beam subjected to distributed loads q_x, q_z, m_y and end loads $\tilde{q}_x, \tilde{q}_z, \tilde{m}_y$ with the corresponding displacements $u_0, w_0, -\theta_0$ as shown in Fig.1. From the Bernoulli-Euler hypothesis, the physical strain is given by

$$\varepsilon_x(z) = \frac{dx'(z) - dx}{dx} = z\kappa_y \tag{1}$$

where κ_y is the generalized strain as

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$$\kappa_y = -\frac{d\theta_0}{dx} \tag{2}$$

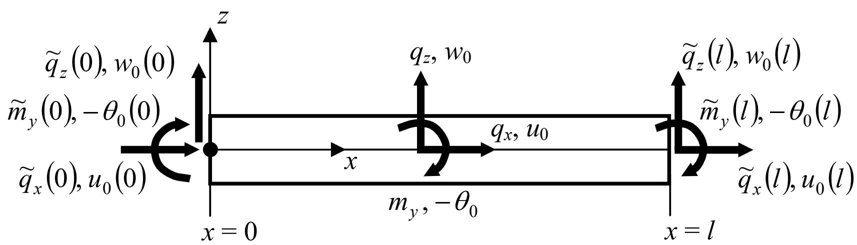

Figure 1. Elastica Subjected to External Loads.

Hooke's law yields the physical stress

$$\sigma_x(z) = E\varepsilon_x(z) \tag{3}$$

and the generalized stress

$$M_y = \iint \sigma_x(z) z\, dA = EI_{zz}\kappa_y \tag{4}$$

The principle of virtual work for the beam is expressed as

$$\int_0^l M_y \delta\kappa_y\, dx - \int_0^l \left(q_x \delta u_0 + q_z \delta w_0 - m_y \delta\theta_0\right) dx \\ -\left[\tilde{q}_x \delta u_0 + \tilde{q}_z \delta w_0 - \tilde{m}_y \delta\theta_0\right]_{x=0,\, l} = 0 \tag{5}$$

From the inextensibility of the elastica, we have

$$\cos\theta_0 = 1 + \frac{du_0}{dx} \quad \sin\theta_0 = \frac{dw_0}{dx} \tag{6}$$

Substituting Eqs.(6) into Eq.(5), we obtain the principle of virtual work for elastica exclusively expressed in terms of the rotation θ_0 [1] as

$$\int_0^l M_y \delta\kappa_y\, dx + \int_0^l \left[-\left\{ \tilde{q}_x(0) + \int_0^x q_x dx' \right\} \sin\theta_0 \right. \\ \left. + \left\{ \tilde{q}_z(0) + \int_0^x q_z dx' \right\} \cos\theta_0 + m_y \right] \delta\theta_0 dx - \left[-\tilde{m}_y\, \delta\theta_0 \right]_{x=0,\, l} = 0 \tag{7}$$

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Finite Element Formulation for Elastica

We consider a two-node element subjected to distributed loads q_x, q_y, m_y and the nodal forces $\tilde{q}_x(0), \tilde{q}_z(0), -M_1$; $\tilde{q}_x(L), \tilde{q}_z(L), -M_2$ as shown in Fig.2.

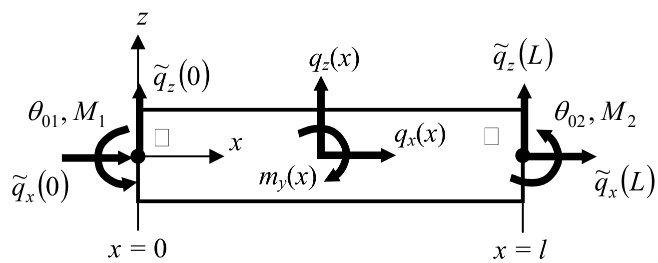

Figure 2.Two-Node Element for Elastica.

The rotation of the element is approximated by the linear shape functions in terms of the nodal displacements ***u*** as

$$\theta_0(x) = [1 - x/L \quad x/L]\,\boldsymbol{u} \qquad \boldsymbol{u}^T = [\theta_{01} \quad \theta_{02}] \tag{8}$$

Introducing Eq.(8) into Eq.(7), we get

$$\begin{aligned}&[\delta\theta_{01} \quad \delta\theta_{02}]\left\{\int_0^L \frac{1}{L}\begin{bmatrix}-1\\1\end{bmatrix}\frac{EI_{zz}}{L}[-1 \quad 1]\,dx\begin{bmatrix}\theta_{01}\\\theta_{02}\end{bmatrix}\right.\\&+\int_0^L\begin{bmatrix}1-x/L\\x/L\end{bmatrix}\left[-\left\{\tilde{q}_x(0)+\int_0^x q_x\,dx'\right\}\sin\theta_0\right.\\&\left.\left.+\left\{\tilde{q}_z(0)+\int_0^x q_z\,dx'\right\}\cos\theta_0+m_y\right]dx-\begin{bmatrix}M_1\\M_2\end{bmatrix}\right\}=0\end{aligned} \tag{9}$$

which gives

$$\boldsymbol{X} = \boldsymbol{K}\,\boldsymbol{u} + \boldsymbol{P} \tag{10}$$

where ***X*** and ***K*** are the nodal forces and the stiffness matrix, respectively,

$$\boldsymbol{X} = \begin{bmatrix}M_1\\M_2\end{bmatrix} \qquad \boldsymbol{K} = \int_0^L \frac{EI_{zz}(x)}{L^2}dx\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} \tag{11}$$

and ***P*** are the equivalent nodal forces due to external loads

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$$\boldsymbol{P} = \int_0^L \begin{bmatrix} 1 - x/L \\ x/L \end{bmatrix} \left[-\left\{ \tilde{q}_x(0) + \int_0^x q_x \, dx' \right\} \sin\theta_0 \right.$$
$$\left. + \left\{ \tilde{q}_z(0) + \int_0^x q_z \, dx' \right\} \cos\theta_0 + m_y \right] dx \tag{12}$$

By assembling all the element stiffness equations given by Eq.(10), we obtain the global stiffness equations as

$$\boldsymbol{R} = \boldsymbol{K} \, \boldsymbol{u} + \boldsymbol{P} \tag{13}$$

which can be solved by the Newton-Raphson method.

Cantilever with a Vertical End Load

We analyze a horizontal cantilever subjected to a vertical end load as shown in Fig.3. The end displacements δ and Δ are shown in Fig.4 as a function of

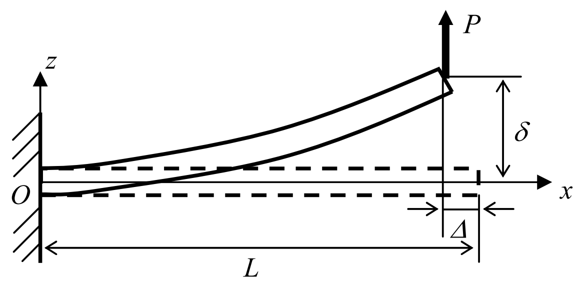

Figure 3 Cantilever with a Vertical End Load.

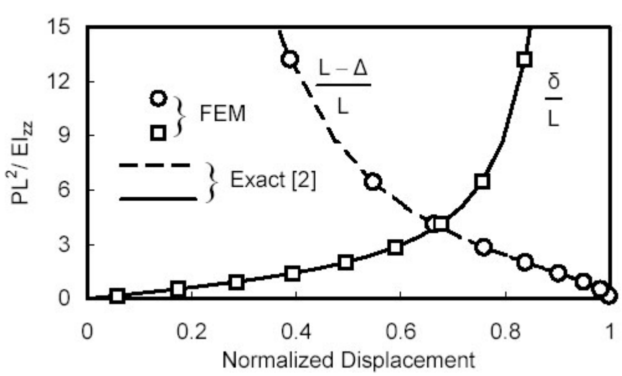

Figure 4. Load-Displacement Relations for Cantilever with a Vertical Load.

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load P together with the closed form solution by the elliptic functions and integrals [2]. It can be seen that the finite element predictions are in good agreement with the exact solution.

Cantilever Subjected to Distributed Compressive Load

We consider the buckling of a horizontal cantilever subjected to uniformly distributed horizontal compressive load $-q_x^o$ as shown in Fig.5. Load-deflection relation is shown in Fig.6 together with the analytical solution by the perturbation method [2] where q_{cr}^o is the buckling load of the beam.

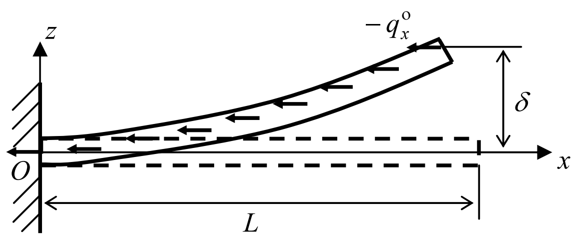

Figure 5. Cantilever Subjected to Distributed Compressive Load.

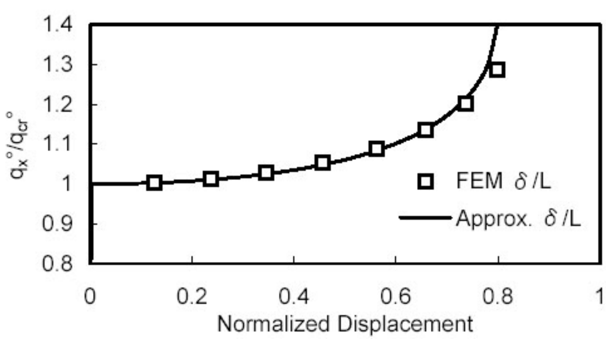

Figure 6. Load-Deflection Relation for Cantilever Subjected to Distributed Compressive Load.

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Tapered Cantilever with Vertical End Load

We analyze a horizontal tapered cantilever subjected to a vertical end load as shown in Fig.7. The end displacements δ and Δ are shown as a function of the load P in Fig.8 together with the analytical solution by the numerical integration [3].

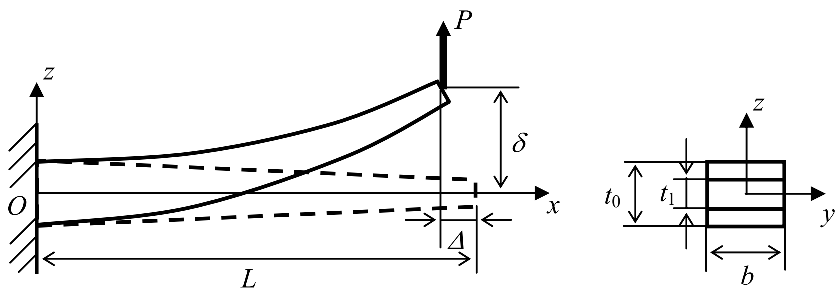

Figure 7. Tapered Cantilever with a Vertical End Load.

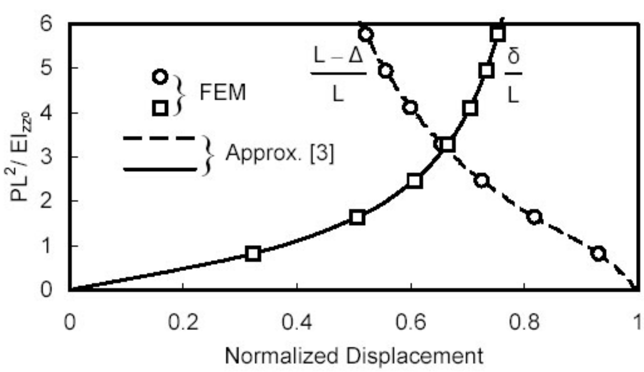

Figure 8. Load-Displacement Relation for Tapered Cantilever with a Vertical End Load (t_1/t_0 =2/3).

References

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2. Frish-Fay, R. (1962): *Flexible Bars*, Butterworths, London.
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