Spectral Element simulation of rupture dynamics along planar and non planar faults

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Summary

A new Spectral Element Method to solve earthquake rupture dynamics along non planar faults has been derived. The method takes implicitely into account non regular contact and frictional conditions on the fault. The method is shown to be quite flexible and accurate. For slip weakening friction law, the method allows to resolve the nucleation phase as well as the dynamic rupture front propagation. Several examples will be presented both for planar and non planar fault geometries.

Introduction

The development of dense digital seismic networks in several seismogenic areas (California, Japan), has bring new constrains of the spectral content of seismic sources, of the slip and slip velocity spatio-temporal distributions during the earthquake rupturing process, and of the nucleation phase of some large earthquakes. Numerical simulation of dynamic earthquake rupturing require todays the resolution of different time and space scales in order to capture the rupture front propagation and the short wave radiation associated with heterogeneous faults of complex geometries.

Two classes of methods are usually adopted to solve faulting problems in seismology: finite differences and boundary integrals. Finite differences, which makke use of a mixed formulation and staggered grids, suffer from grid dispersion errors and spurious diffractions due to the discretization, when modeling curved faults [1]. On the other hand, boundary integrals are able to model 3D non coplanar fault segments but are limited to homogeneous elastic media [2].

The spectral element method (SEM) combines the geometrical flexibility of the finite elements with the exponential convergence rate of the spectral methods [3]. This method has been recognized to be a powerful tool for the simulation of wave propagation at local and global scales [4]. Since SEM is based on a variational approximation of the elastodynamics equation and a piecewise polynomial approximation, frictional and contact conditions along non planar fault interfaces can be naturally taken into account in contrast to finite differences, which usually rely on a stress-velocity staggered discretization, where mixed fault boundary conditions can only be handled through some interpolation. In the latter case, even for simple planar fault geometries, a large number of points is required in order to provide the same spatial resolution as SEM, .

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Problem statement

Let us consider an elastic body of volume $\Omega \subseteq \mathbb{R}^{n_d}$($n_d$ is the geometrical dimension, and $n_d = 2,3$ for geophysical applications), whose surface consists of an external surface $\partial\overline{\Omega}$, on which homogeneous boundary conditions are applied, and an internal surface $\Gamma(\mathbf{x})$, accounting for the presence of a fault. Fault boundary conditions will involve, indeed, potential discontinuities in the kinematic fields (displacement, velocity and acceleration) across Γ. Ω is assumed to be decomposed into two non overlapping subdomains Ω_1 and Ω_2, with a common interface $\gamma \supseteq \Gamma$, such that $\mathbf{n}$ is the external normal to Ω_2 along Γ. The restriction of the displacement and velocity to Ω_i is denoted as $\mathbf{u}_i(\mathbf{x},t)$ and $\mathbf{v}_i(\mathbf{x},t)$ respectively, for any $\mathbf{x} \in \Omega_i$ and $t \in I = [0,T]$. For any field $\mathbf{w}$, the discontinuity of $\mathbf{w}$ across Γ is defined as $\delta\mathbf{w} = \mathbf{w}_2|_\Gamma - \mathbf{w}_1|_\Gamma$.

The elastodynamic equation, governing the system in Ω_i, is:

$$\rho\frac{\partial \mathbf{v}_i}{\partial t} = \nabla\cdot\boldsymbol{\sigma}_i + \mathbf{f}_i; \quad \boldsymbol{\sigma}_i = \mathbf{c} : \nabla\mathbf{u}_i \tag{1}$$

where ρ is the density, $\mathbf{f}$ is the resultant of the external forces by unit volume, $\mathbf{c}$ is the fourth-order elastic coefficients tensor, and $\boldsymbol{\sigma}$ is the stress tensor.

Either a free surface ($\boldsymbol{\sigma}_i\cdot\mathbf{n} = \mathbf{0}$) or a rigid boundary ($\mathbf{u}_i = \mathbf{0}$) are assumed as boundary conditions on the $\partial\overline{\Omega}\cap\partial\Omega_i$, and the system is supposed to be at rest before $t = 0$, so that initial conditions are $\mathbf{u}_i(\mathbf{x},0) = \mathbf{v}_i(\mathbf{x},0) = \mathbf{0}$ in Ω_i. Continuity of displacements ($\delta\mathbf{u} = \mathbf{0}$) and tractions ($\delta(\boldsymbol{\sigma}\cdot\mathbf{n}) = \mathbf{0}$) may be imposed on $\gamma - \Gamma$, while two simple physical constraints can be verified on Γ, through a contact condition and a frictional condition.

Let us define the total reaction $\mathbf{R}_i^T$ on the fault as the sum of a static contribution $\mathbf{R}_i^0$ and a dynamic perturbation $\mathbf{R}_i = -\boldsymbol{\sigma}\cdot\mathbf{n}$, coming from the elastic bulk. Because of the action-reaction principle, $\mathbf{R}_1^T = -\mathbf{R}_2^T \equiv -\mathbf{R}^T$. Any vector field $\tilde{\mathbf{w}}$, defined on the fault, may be decomposed onto its tangential and normal components $\tilde{\mathbf{w}} = \tilde{\mathbf{w}}_t + \tilde{w}_n\mathbf{n}$. The contact condition (or Signorini law) requires that either the two sides of the fault are stuck, or, if opening occurs, both sides act like a free surface. Formally, this yields:

$$\delta u_n \geq 0 \qquad ; \qquad R_n^T \leq 0 \qquad ; \qquad \delta u_n R_n^T = 0 \tag{2}$$

The frictional condition (Coulomb law) requires that the system is at rest, until the tangential reaction reaches a threshold value, proportional to the normal reaction. While sliding, the tangential reaction is constrained to the threshold, and the slip rate is colinear with the tangential reaction. The formal statement of this condition is:

$$\Phi(R_t^T, R_n^T) := \left(R_t^T + \mu R_n^T\right) \leq 0 \quad ; \quad \delta v_t = \dot{\gamma}\frac{R_t^T}{\|R_t^T\|} \quad ; \quad \dot{\gamma} \geq 0 \quad ; \quad \dot{\gamma}\Phi = 0 \tag{3}$$

where R_t^T and δv_t are the absolute values of the tangential reaction and slip rate respectively, and μ is the frictional coefficient. In seismology, a linear slip weakening law [5] i s generally assumed.

Variational formulation

A variational statement of the elastodynamic problem is introduced. The space of the admissible displacements and velocities is $\mathcal{H}^1 = H^1(\Omega_1) \times H^1(\Omega_2) \times I$. A solution is obtained by providing a couple $(\mathbf{u}, \mathbf{v}) \in \left(\mathcal{H}^1\right)^{n_d} \times \left(\mathcal{H}^1\right)^{n_d}$, satisfying the equation:

$$(\rho \dot{\mathbf{v}}, \mathbf{w}) + a(\mathbf{u}, \mathbf{w}) = (\mathbf{f}, \mathbf{w}) - \langle \mathbf{R}, \mathbf{w} \rangle \tag{4}$$

for any test function $\mathbf{w} \in \mathcal{H}_0^1 = H^1(\Omega_1) \times H^1(\Omega_2)$. The above expressions in parenthesis are explicitly written as follows:

$$(\mathbf{v}, \mathbf{w}) = \sum_i \int_{\Omega_i} \mathbf{v}^i \cdot \mathbf{w}^i d\Omega;\ a(\mathbf{u}, \mathbf{w}) = \sum_i \int_{\Omega_i} \nabla \mathbf{w}^i : \boldsymbol{\sigma}\left(\mathbf{u}^i\right) d\Omega;\ \left\langle \mathbf{R}, \mathbf{w}^i \right\rangle = \sum_i \int_{\gamma} \mathbf{w} \cdot \mathbf{R} d\gamma \tag{5}$$

Spectral element approximation

Spectral element methods solve the variational statement of the elastodynamic equation by covering both domains Ω_i with hexahedral elements Ω_i^e, such that $\Omega_i = \bigcup_{e=1}^{N_i} \Omega_i^e$, and by approximating the spaces $\mathcal{H}^1$ and $\mathcal{H}_0^1$ with the subspaces $\mathcal{H}^h$ and $\mathcal{H}_0^h$ of finite dimension, for which a systematic derivation of basis functions is provided.

Hexahedra are obtained from a reference element $\Lambda = [-1,1]^{n_d}$, through an invertible map $\mathcal{F}_e : \Lambda \rightarrow \Omega_e$ local in each element. Thus, the finite dimensional subspaces may be defined as follows:

$$\mathcal{H}_t^h = \{\mathbf{g}^h(\mathbf{x}, t) \in \mathcal{H}_t^1 \ : \ \forall t, \mathbf{g}^h|_e \circ \mathcal{F}_e \in \text{span}\{\mathbb{P}_N(\Lambda)\}^{n_d}\} \tag{6}$$

and

$$\mathcal{H}_0^h = \{\boldsymbol{\phi}^h(\mathbf{x}) \in \mathcal{H}_0^1 \ : \ \boldsymbol{\phi}^h|_e \circ \mathcal{F}_e \in \text{span}\{\mathbb{P}_N(\Lambda)\}^{n_d}\} \tag{7}$$

Here $f^h|_e$ is the restriction of the function f to the element Ω_i^e and $\mathbb{P}_N$ are polynomials of degree equal or less than N.

The polynomial subspace can be defined through one of its bases. Let us consider the zeroes of Gauss-Lobatto-Legendre (GLL) inside the interval $[-1,1]$, and the Lagrange interpolating function associated to this set of points. For $N+1$ GLL points, maximum

polynomial degree is N; since Lagrange polynomials are independent, they go to make up a basis for $\mathbb{P}_N([-1,1])$. A basis for $\mathbb{P}_N(\Lambda)$, indeed, can be obtained by tensorization of 1D Lagrange polynomials. Integrals, implied in the formulation, can be referred to the single hexahedra and then, they can be mapped back onto the reference element.

Quadrature formula using GLL points leads finally to a linear system:

$$\mathbf{M}_i\dot{\mathbf{V}}_i = \mathbf{F}_i^{ext} + \mathbf{F}_i^{int}(\mathbf{U}_i) - \mathcal{B}^T\mathbf{R}_i \tag{8}$$

Here $\mathbf{U}_i$ and $\mathbf{V}_i$ are respectively the values of the displacement and velocity, component by component, at the collocation points, inside the domain Ω_i, $\mathbf{M}_i$ are the mass matrices and thay are diagonal. Finally, the matrix $\mathcal{B}^T$ comes from the boundary integral. Since it accounts only for the geometrical properties of the common interface, it is independent of the domain index i, if a matching mesh between Ω_1 and Ω_2 is assumed.

Let us reduce the discrete equation on the fault:

$$\tilde{\mathbf{M}}_i\dot{\mathbf{V}}_i^f = \tilde{\mathbf{F}}_i - \mathbf{B}^T\mathbf{R}_i \tag{9}$$

If $\mathbf{H}_i$ is the projection operator onto the fault, then $\mathbf{B}^T = \mathbf{H}\mathcal{B}^T$, $\tilde{\mathbf{M}}_i = \mathbf{H}_i\mathbf{M}_i\mathbf{H}_i^T$, $\mathbf{V}_i^f = \mathbf{H}_i\mathbf{V}_i$ and $\tilde{\mathbf{F}}_i = \mathbf{H}_i\left(\mathbf{F}_i^{ext} + \mathbf{F}_i^{int}(\mathbf{U}_i)\right)$. Since $\mathbf{H}_i$ operator extracts some blocks from the matrices, to which it is applied, $\tilde{\mathbf{M}}_i$ matrices inherit diagonality and invertibility from their parents $\mathbf{M}_i$. Finally discontinuity of velocity on the fault can be obtained by subtraction:

$$\delta\dot{\mathbf{V}}_i^f = \delta\dot{\mathbf{V}}^{free} - \left(\tilde{\mathbf{M}}_2^{-1} + \tilde{\mathbf{M}}_1^{-1}\right)\mathbf{B}^T\mathbf{R} \quad ; \quad \delta\dot{\mathbf{V}}^{free} = \left(\tilde{\mathbf{M}}_2^{-1}\tilde{\mathbf{F}}_2 - \tilde{\mathbf{M}}_1^{-1}\tilde{\mathbf{F}}_1\right) \tag{10}$$

Time evolution

A time evolution of the system through a Newmark velocity scheme can be introduced inside the single domains:

$$\mathbf{V}_i^{(n+1)} = \mathbf{V}_i^{(n)} + \Delta t\,\mathbf{M}_i^{-1}\left(\mathbf{F}_i^{int}\left(\mathbf{U}_i^{(n+1/2)}\right) + \mathbf{F}_i^{ext\,(n+1/2)}\right) \tag{11}$$

$$\mathbf{U}_i^{(n+1)} = \mathbf{U}_i^{(n)} + \frac{1}{2}\Delta t\,\mathbf{V}_i^{(n)} + \frac{1}{2}\Delta t\,\mathbf{V}_i^{(n+1)} \tag{12}$$

where n is the time iteration, associated to a time step Δt. It is worth to note that both equations are centered around $t_{n+1/2}$. The relation involving the discontinuity of velocity and the reaction on the fault inherits the same centered scheme:

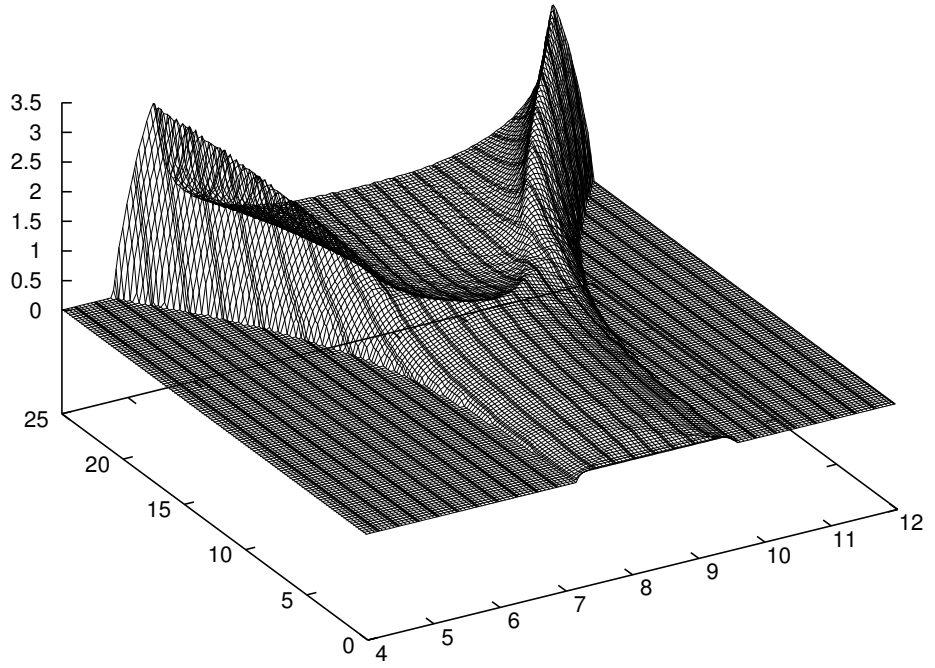

Figure 1: Slip velocity as function of time and position on the fault. When the crack progesses slowly, during the nucleation, stopping phases are seen coming back into the active zone, influencing the dynamics

$$\delta\mathbf{V}^{(n+1)} = \delta\mathbf{V}^{(n)} + \Delta t\,\delta\dot{\mathbf{V}}^{free\,(n+1/2)} - \frac{1}{2}\mathbf{C}_\Gamma\mathbf{R}^{(n)} - \frac{1}{2}\mathbf{C}_\Gamma\mathbf{R}^{(n+1)} \tag{13}$$

where $\mathbf{C}_\Gamma = \Delta t\,\left(\tilde{\mathbf{M}}_2^{-1} + \tilde{\mathbf{M}}_1^{-1}\right)\mathbf{B}^T$ The above equation is a global relation, involving all the collocation points on the fault. However, because of the diagonality of $\mathbf{C}_\Gamma$, it can also be referred to the single collocation point. At that stage, the reactions can be solved as the intersection between this affine relation and the Signorini's and Coulomb's graphs. Once the admissible reaction has been determined, it is injected back into the bulk and propagated. The non regularity of the contact conditions requires to derive a consistent high frequency regularization.

Fault dynamics

Let us consider a simple 2D example to describe the evolution of the rupture along a curved fault. The latter cuts the elastic bulk into two non overlapping domains, whose borders are perfeclty matched layers. On the fault, homogeneous pre-stress and frictional values are assumed. Hence, the rupture is initialized by raising up the intial traction to the

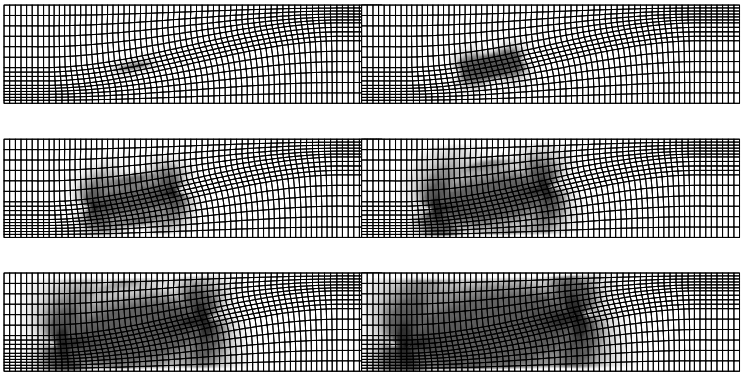

Figure 2: Snapshots of the velocity modulus for a spontaneously propagating crack. The fault line is individuated by a thicker black line. Because of curvature of the fault, a rotation effect in the field occurs behind the crack tip.

frictional threshold in a patch enough larger to allow the crack to evolve. During the nucleation, stopping phases influence the crack dynamics by reducing the initial exponential growth. At the begininning of the propagation, crack progresses at Rayleigh velocity . At same time, the S-peak in the reaction increases its amplitude, till to reach the threshold value. In this case, the rupture jumps ahead of the front tip and it accelerates at supershear speeds. In addition, in curved faults the symmetry on the normal reaction is broken allowing rotation effects to take place behind the crack tip.

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