Crack Front Waves on a Shear Crack

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Summary

The existence of "crack front waves" on the edge of a crack propagating in "opening" mode was rigorously established by making use of a general solution for the perturbation of the crack's edge, developed by J.R. Willis and A.B. Movchan. Corresponding perturbation formulae for a crack under shear loading are here exploited to investigate the possibility of crack front waves on a shear crack. In contrast to the opening mode case, crack front waves do not exist for all crack speeds, but only for speeds above a certain threshhold that depends upon the mixture of Modes II and III.

Introduction

Crack front waves were discovered during a computation by Rice and Morrissey [1] and their existence was confirmed analytically [2] by developing a "dispersion relation" from a general solution [3] for the dynamic perturbation of a crack edge under Mode I loading, in combination with the Griffith energy balance, linearised with respect to the perturbation. This demonstrated that perturbations of the crack edge can propagate without dispersion or attenuation, with a speed v along the edge that depends on the speed V of propagation of the crack. The resultant speed of the trace of the disturbance through the body, $(V^2+v^2)^{1/2}$, likewise varies; it is close to but definitely less than the speed of Rayleigh waves. Subsequent work [4] demonstrated that crack front waves do, in fact, display some dispersion, the solution given in [2] having neglected a term that becomes unimportant as frequency and wavenumber tend to infinity; [4] developed the leading-order "finite-frequency" correction and the corresponding leading-order contribution to attenuation due to viscoelasticity. The present work considers the propagation of a perturbation of the edge of a crack when it is loaded in shear. The basic perturbation solution has been given by Movchan and Willis [5]. It is relatively more complicated than the opening mode case because there is inevitably coupling between Modes II and III. Implications of the Griffith energy balance are analysed, as in [2], in the high-frequency limit, though finite-frequency corrections could certainly be obtained. The waves that are possible depend on the ratio of Modes II and III in the original loading of the unperturbed crack.

The perturbation solution

The configuration of concern is that of a crack, occupying the region defined by

$$-\infty < x_1 < Vt + \varepsilon\phi(x_2,t), \quad -\infty < x_2 < \infty, \quad x_3 = 0. \tag{1}$$

The medium through which the crack propagates is uniform, isotropic and elastic, with Lamé moduli λ, μ and density ρ; the speeds of longitudinal and shear waves are denoted a

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and b respectively, where $a^2 = (\lambda + 2\mu)/\rho$, $b^2 = \mu/\rho$. It is loaded in such a way that the stress field σ_{ij}^A would be generated if the crack were not present. This stress field is taken to be independent of the coordinate x_2 and to depend on (x_1, x_3, t) only in the combination $(x_1 - Vt, x_3)$, so that the unperturbed motion of the crack ($\varepsilon = 0$) is possible. The presence of the crack induces *additional* stress and displacement fields, σ_{ij} and u_i, which satisfy the equations of elastodynamics, the boundary conditions

$$\sigma_{i3} + \sigma_{i3}^A = 0 \tag{2}$$

on the crack surfaces, and a "radiation condition" that this field is composed only of waves travelling outwards from the crack.

Movchan and Willis [5] developed expressions for the perturbations, to first order in ε, of the stress intensity factors as follows.

$$\Delta\mathbf{K} = \varepsilon\{\mathbf{Q}^T * (\phi\mathbf{K}) - \mathbf{E}\phi'\mathbf{K} + (\pi/2)^{1/2}\phi\mathbf{M} - \phi(\mathbf{Q}^T * \mathbf{K})\}. \tag{3}$$

Here, $\mathbf{K}$ is a column vector $(K_{II}, K_{III})^T$, and $\Delta\mathbf{K}$ and $\mathbf{M}$ are similarly column vectors. $\mathbf{Q}$ is a 2×2 matrix-valued function of (x_2, t), and the symbol $*$ denotes convolution with respect to these variables. $\mathbf{E}$ is the skew matrix

$$\mathbf{E} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4}$$

The applied loading generates stresses for the unperturbed crack that take the form, at small distance X from the edge of the crack and in its plane,

$$\begin{pmatrix} \sigma_{13} \\ \sigma_{23} \end{pmatrix} \sim \mathbf{K}(2\pi X)^{-1/2} - \mathbf{P} + \mathbf{M}X^{1/2}. \tag{5}$$

For the loading considered, $\mathbf{K}$, $\mathbf{P}$ and $\mathbf{M}$ are constants[2], and the speed of the crack is such that the Griffith energy balance

$$\mathbf{K}^T A(V)\mathbf{K} = G_c \tag{6}$$

is satisfied, where $A(V)$ is a diagonal matrix with entries given in [6], for instance. The matrix-valued function $\mathbf{Q}$ is given in [5]; it is obtained from a near-crack-tip expansion of the dynamic Mode II/III weight function, itself obtained from the explicit solution of a matrix Wiener–Hopf problem. The formulae are lengthy and are not repeated here.

[2] In the case of general loading, these coefficients would depend on x_2 and t and the coefficient of $X^{1/2}$ would become $\mathbf{M} - (2/\pi)^{1/2}\mathbf{Q}^T * \mathbf{K}$. Equation (3) is applicable to the general case.

Equation governing ϕ

For the perturbed solution, the Griffith energy balance (6) must remain satisfied, at each point of the crack edge, with $\mathbf{K}$ replaced by $\mathbf{K}+\Delta\mathbf{K}$ and V replaced by $(V+\varepsilon\,\dot{\phi})(1+\varepsilon^2\phi_{,2}^2)^{-1/2}$. Expanded to first order in ε, this requires that

$$2\mathbf{K}^T A(V)\Delta\mathbf{K}+\varepsilon\dot{\phi}\mathbf{K}^T A'(V)\mathbf{K}=0, \tag{7}$$

where $\Delta\mathbf{K}$ is given in terms of ϕ by equation (3). Now seeking a solution proportional to $\exp[-i(\omega t+kx_2)]$, ω and k have to be related so as to satisfy the dispersion relation that comes from Fourier transforming (7), together with (3):

$$2(\mathbf{K}^T A\tilde{\mathbf{Q}}^T\mathbf{K}+ik\mathbf{K}^T\mathbf{E}\mathbf{K})-i\omega\mathbf{K}^T A'\mathbf{K}+(\pi/2)^{1/2}\mathbf{M}=0, \tag{8}$$

where $\tilde{\mathbf{Q}}$ represents the Fourier transform of $\mathbf{Q}$, given explicitly in [5]. In fact, $\tilde{\mathbf{Q}}$ is a homogeneous function of degree 1 in (k,ω) and therefore the term involving $\mathbf{M}$ can be neglected at high frequency. The relation (8) then becomes a non-dispersive equation for the wave speed $v=-\omega/k$. Remarkably, it is possible to *prove* that the imaginary part of the left side of (8) vanishes identically, for all real v such that $(v^2+V^2)^{1/2}$ is less than the speed of Rayleigh waves. Therefore, it is only necessary to plot its real part as a function of v, and see whether it is ever zero.

Results

The real part of (8), with $\mathbf{M}\to 0$ and $v=-\omega/k$, can be expressed in the form

$$\frac{1}{4}\left[K_{II}^2 A_{II}(V)P_{II}(v)+K_{III}^2 A(V)P_{III}(v)\right]+i\operatorname{sgn}(v)K_{II}K_{III}\frac{(a_+\overline{b_+}-\overline{a_+}b_+)}{(a_+a_-+b_+b_-)}=0. \tag{9}$$

The terms are not defined in detail here, but $P_{II}(v)=0$ would describe a crack front wave under pure Mode II loading, and $P_{III}=0$ would define a Mode III crack front wave. The corresponding relation defining the Mode I crack front wave is denoted

$$P_I(v)=0. \tag{10}$$

Figure 1 shows plots of P_I, P_{II} and P_{III}, for the case $V=0.2b$. For all of the results presented here, $a^2/b^2=3$, corresponding to a Poisson's ratio of $1/4$. The graph of P_I crosses the axis, but those of P_{II} and P_{III} do not. Thus, there is a crack front wave, whose speed v is about $0.86b$, for a crack under Mode I loading, but no such wave for pure Mode II or pure Mode III loading, for this crack speed.

Figure 2, however, shows plots for $V=0.8b$. This time, there *is* a Mode II crack front wave, for which $v\approx 0.28b$ corresponding to $(v^2+V^2)^{1/2}\approx 0.848b$. The corresponding

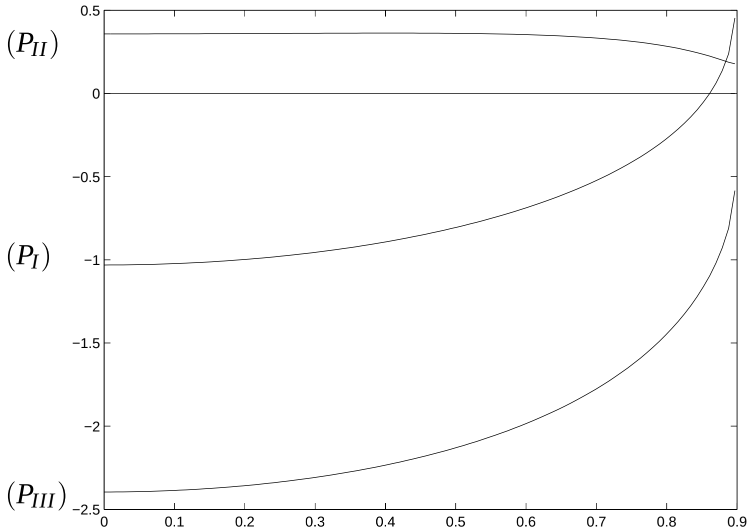

Figure 1: The functions $P_I(v)$, $P_{II}(v)$ and $P_{III}(v)$, plotted against v/b, for the case $V/b = 0.2$.

Rayleigh wave speed is $0.9194b$. The Mode I crack wave speed is $v \approx 0.43b$, so that $(v^2 + V^2)^{1/2} \approx 0.908b$. The crack speed V below which there is no Mode II crack front wave is approximately $0.715b$.

There is never, in fact, a pure Mode III crack front wave. However, since $P_{III} < 0$, there is the possibility of finding crack front waves under mixed Modes II and III loading, at crack speeds for which no pure Mode II crack front wave exists. Figure 3 shows an example for which $K_{II} = 1$ and $K_{III} = 0.4$, with $V = 0.5b$. There is a crack front wave, for which $v \approx 0.58b$. It is interesting also to note that the speed of the wave is sensitive to direction, since equation (8) is unchanged if the signs of both $K_{II}K_{III}$ and v are changed. There is, in fact, no crack front wave with $v > 0$ when $K_{II} = 1$, $K_{III} = -0.4$ and $V = 0.5b$.

Reference

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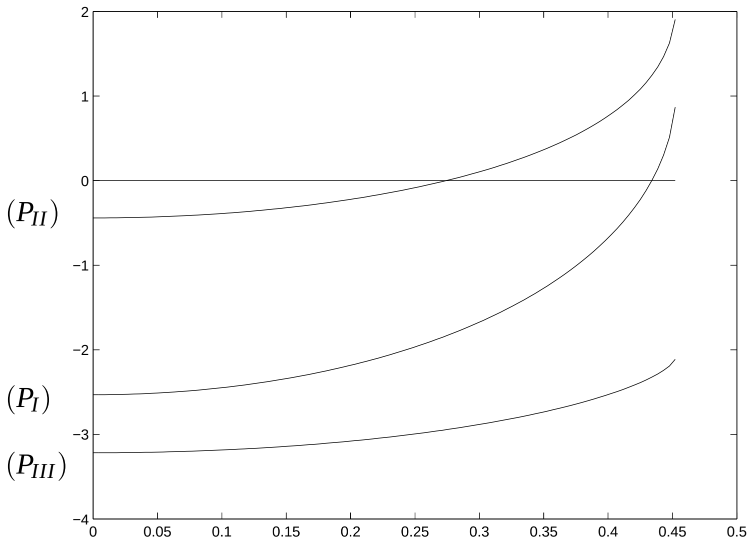

Figure 2: The functions $P_I(v)$, $P_{II}(v)$ and $P_{III}(v)$, plotted against v/b, for the case $V/b = 0.8$.

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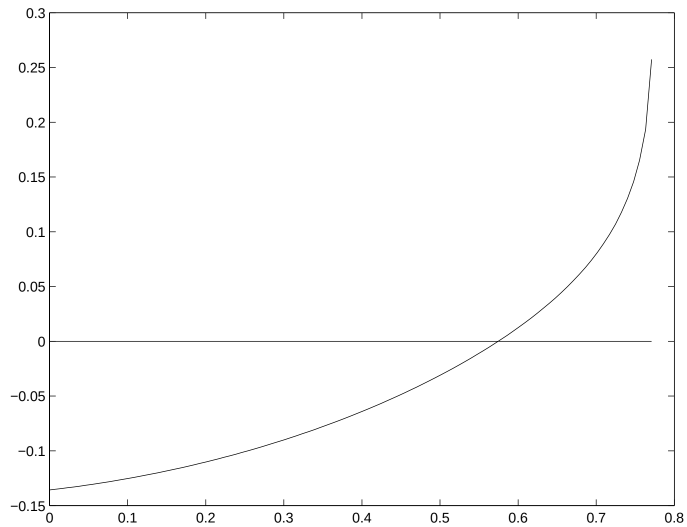

Figure 3: Plot of the left side of equation (8), for $K_{II} = 1.0$ and $K_{III} = 0.4$ against v/b, for the case $V/b = 0.5$.