Numerical determination of eigenfrequencies and eigenmodes using the Method of Fundamental Solutions

Carlos J. S. Alves[1], Pedro R. S. Antunes[1]

Summary

In this work we show the application of the Method of Fundamental Solutions (MFS) in the determination of eigenfrequencies and eigenmodes associated to wave scattering problems. This meshless method has already been applied to simple geometry domains with Dirichlet boundary conditions (e.g. [8]). Here we show that a particular choice of point-sources can lead to very good results for a fairly general type of domains. Simulations with other types of boundary conditions are also considered.

Introduction

The determination of the eigenvalues and eigenfunctions associated to the Laplace operator in a bounded domain Ω is a well known problem with applications in acoustics (e.g. [5], [6]). For simple shapes, such as rectangles or circles in 2D, this leads to straightforward computations, without the need of a numerical algorithm. However, when the shape is non trivial, that computation requires the use of a numerical method for PDEs. A standard finite differences method can produce good results when dealing with a particular type of shapes defined on rectangular grids, while for other type of shapes the finite element method or the boundary element method are more appropriated (e.g. [10]). These classical methods require extra computational effort; in one case, the construction of the mesh and the associated rigid matrix, and in the other, the integration of weakly singular kernels. Here we propose a meshless method for solving the eigenvalue problem using the method of fundamental solutions (MFS). The MFS has been mainly applied to boundary problems in PDEs, starting in the 1960s (e.g. [9] or [2]). An account of the development can be found in [7]. The application of the MFS to the calculation of the eigenvalues has been introduced by Karageorghis in [8], and applied for simple shapes. In [8] it is presented a comparison with the boundary element method used by De Mey in [10], and the results obtained for simple shapes (circles, squares), show a better performance for the MFS. The application of other meshless methods to the determination of eigenfunctions and eigenmodes has also been subject to recent research, mainly using radial basis functions (e.g. [4]).
In this work we consider the application of the MFS to general shapes. In that case the choice of the source points in the MFS becames more important to retrieve higher eigenfrequencies. We are able to obtain good results with a particular algorithm associating the source points to the shape. Having determined an approximation of the eigenvalue, we apply an algorithm based on the MFS to obtain the associated eigenmodes.

Helmholtz equation

Let $\Omega \subset \mathbb{R}^2$ be a bounded connected domain with regular boundary $\partial\Omega$. For simplicity we will consider the 2D - Dirichlet eigenvalue problem for the Laplace operator. This is

[1]CEMAT, Department of Mathematics, Instituto Superior Técnico, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal. (calves@math.ist.utl.pt; pant@math.ist.utl.pt)

equivalent to obtain the resonance frequencies κ that verify the Helmholtz equation

$$\begin{cases} \Delta u + \kappa^2 u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

for a non null function u. As an application, this corresponds to recover the resonance frequencies $\kappa > 0$ associated with a particular shape of a drum Ω.
A fundamental solution Φ_κ of the Helmholtz equation verifies $(\Delta + \kappa^2)\Phi = -\delta$, where δ is the Dirac delta distribution. In the 2D case, we take

$$\Phi_\kappa(x) = \frac{i}{4} H_0^{(1)}(\kappa|x|) \tag{2}$$

where $H_0^{(1)}$ is the first Hänkel function.

A density result in [1] states that if κ is not an eigenfrequency then

$$L^2(\partial\Omega) = \overline{\text{span}\left\{\Phi_\kappa(x-y)|_{x\in\partial\Omega} : y \in \hat{\Gamma}\right\}}, \tag{3}$$

where $\hat{\Gamma}$ is an admissible source set, for instance, the boundary of a bounded open set $\hat{\Omega} \supset \overline{\Omega}$, considering $\hat{\Gamma}$ surrounding $\partial\Omega$. This allows to justify the approximation of a L^2 function, with complex values, defined on $\partial\Omega$, using a sequence of functions

$$u_m(x) = \sum_{j=1}^{m} \alpha_{m,j} \Phi_\kappa(x - y_{m,j}) \tag{4}$$

that converges to $u|_\Gamma$ in $L^2(\partial\Omega)$. This is a partial justification to the convergence of the Method of Fundamental Solution (MFS) based on density results. It is similar to Bogomolny's approach in [3], but it avoids the use of boundary layer potentials. As pointed out in [1] or [3], the convergence of the MFS, in a general case, is not completely related to the discretization of a single layer potential, although there is a straightforward relation. A single layer potential defined on $\hat{\Gamma}$ is an analytic function in Ω, and therefore such an approach would only be appropriate for analytic functions.

Since $u|_\Gamma \equiv 0$ is an analytic function, it makes sense to consider the approach of the MFS as being related to the discretization of the single layer potential, for $x \notin \hat{\Gamma}$,

$$\mathcal{S}_\kappa\varphi(x) = \int_{\hat{\Gamma}} \Phi_\kappa(x-y)\varphi(y)\, ds_y \approx u_m(x) = \sum_{j=1}^{m} \alpha_{m,j}\Phi_\kappa(x - y_{m,j}). \tag{5}$$

Theorem:
If κ is not an eigenfrequency of the interior Dirichlet problem then $\dim(Ker(\mathcal{S}_\kappa)) = 0$.
Proof. If κ is not an eigenfrequency then $\mathcal{S}_\kappa\varphi = 0$ on $\partial\Omega$ implies $\mathcal{S}_\kappa\varphi = 0$ in Ω, by the well posedness of the interior Dirichlet problem. Using the analyticity of $\mathcal{S}_\kappa\varphi$, this implies $\mathcal{S}_\kappa\varphi = 0$ in $\hat{\Omega}$ and the continuity of the traces implies $(\mathcal{S}_\kappa\varphi)^+ = (\mathcal{S}_\kappa\varphi)^- = 0$ on $\hat{\Gamma}$. Therefore, by the well posedness of the exterior Dirichlet problem, with the Sommerfeld radiation condition (verified by $\mathcal{S}_\kappa\varphi$), this implies $\mathcal{S}_\kappa\varphi = 0$ in $\mathbb{R}^2$. In conclusion, $\mathcal{S}_\kappa\varphi = 0$ on $\partial\Omega$ implies $\varphi = 0$, and therefore $\dim(Ker(\mathcal{S}_\kappa)) = 0$. □

Thus, using this result, we search for κ such that $\dim(Ker(\mathcal{S}_\kappa)) \neq 0$.

Numerical Algorithm

From the previous considerations we may sketch a procedure of finding the eigenvalues by checking the frequencies κ for which $\dim(Ker(\mathcal{S}_\kappa)) \neq 0$. Defininning m collocation points $x_i \in \partial\Omega$ and m source points $y_{m,j} \in \hat{\Gamma}$,we obtain the system

$$\sum_{j=1}^{m} \alpha_{m,j} \Phi_\kappa(x_i - y_{m,j}) = 0, \quad (x_i \in \partial\Omega). \tag{6}$$

Therefore a straighforward procedure is to find the values κ for which the $m \times m$ matrix

$$A(\kappa) = [\Phi_\kappa(x_i - y_j)]\,[\alpha_j]_{m\times m} \tag{7}$$

has a null determinant. However, an arbitrary choice of source points may lead to worst results than the expected with the MFS applied to simple shapes. We will choose the points $x_1, ..., x_m \in \partial\Omega$ and $y_1, ..., y_m \in \hat{\Gamma}$ in a particular way. Given the m points x_i on $\partial\Omega$, we take m point sources

$$y_i = x_i + \tilde{\mathbf{n}}_i$$

where $\tilde{\mathbf{n}}_i$ is approximately normal to the boundary $\partial\Omega$ on x_i. To obtain the vector $\tilde{\mathbf{n}}_i$ we just consider $\tau^- = x_i - x_{i-1}$, $\tau^+ = x_i - x_{i+1}$ and calculate $\mathbf{n}^-$, $\mathbf{n}^+$ which are normal to τ^- and τ^+ (respectively) and pointing outwards Ω. Then we take $\tilde{\mathbf{n}}_i = \frac{1}{2}(\mathbf{n}^- + \mathbf{n}^+)$. By some experimental criteria, we will usually take $|\tilde{\mathbf{n}}_i| = \beta(\approx 1/3)$. Source points taken too far from the boundary only presented better results in some particular cases, using simple shapes.

The components of the matrix $A(\kappa)$ are complex numbers, so the determinant is also a complex number. We consider the real function $g(\kappa) = |Det[A(\kappa)]|$. It is clear that the function g will be very small in any case, since the MFS is highly ill conditioned and the determinant is quite small. To avoid machine precision problems the code was built in *Mathematica*.
If κ is an eigenfrequency, κ is a point of minimum where $g(\kappa) = 0$ and therefore the derivative changes sign. We will make use of the rough approximation $g'(w) \approx \frac{g(w+\varepsilon)-g(w-\varepsilon)}{2\varepsilon}$ for a small $\varepsilon > 0$. To approximate the eigenfrequencies, where a clear change on the sign of the derivative g' is attained, we used the simple bissection method, which revealed to be quite accurate in the search of high eigenfrequencies, which are closer to each other.

Once we have an eigenfrequency determined, we may get the eigenfunctions just by considering extra collocation points inside the domain. Depending on the multiplicity of the eigenvalue, we will add one or more collocation points to make the linear system well determined.

Determination of an Eigenmode

To obtain an eigenfunction associated with a certain resonance frequency ω we use a collocation method on $n+1$ points, with $x_1, \cdots, x_n$ on $\partial\Omega$ and a point $x_{n+1} \in \Omega$. Then, the

approximation of the eigenfunction is given by

$$\tilde{u}(x) = \sum_{k=1}^{n+1} \alpha_k \Phi_\omega(x - y_k). \tag{8}$$

To exclude the solution $\tilde{u}(x) \equiv 0$, the coefficients α_k are determinated by the resolution of the system

$$\begin{cases} \tilde{u}(x_i) = 0 & i = 1, \ldots, n \\ \tilde{u}(x_{n+1}) = 1 & , \end{cases} \tag{9}$$

When we take $n = m$ this resumes to add one line and one column to the matrix $A(\kappa)$ defined in (7).

Simulation 1: Dirichlet boundary condition.

We consider a domain Ω_1 with a non trivial boundary given by the parametrization

$$t \mapsto \left(\cos(t) - \frac{\cos(t)\sin(2t)}{2}, \sin(t) + \frac{\cos(4t)}{6} \right).$$

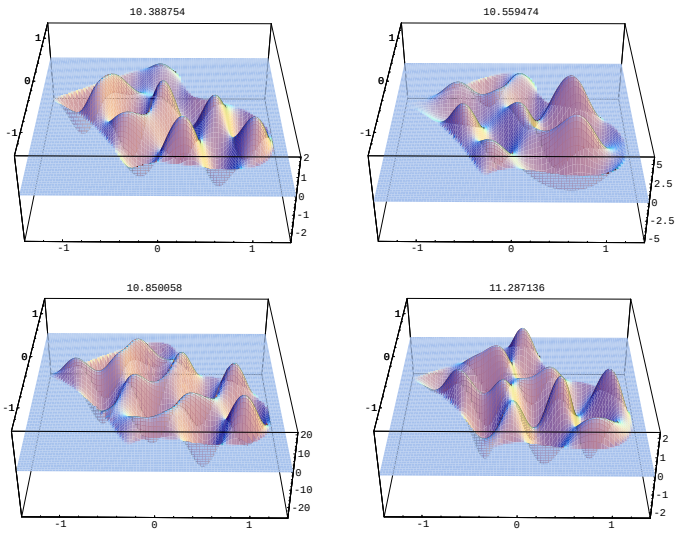

Figure 1: Plots of the 21th$\cdots$24th eigenmodes associated to Ω_1

In Fig. 1 we present 4 plots with the eigenfunctions associated to the 21th$\cdots$24th eigenvalues. In top of each picture it is written the associated eigenfrequency.

In Fig. 2 we show the nodal domains, i.e. the components where the eigenmode keeps the same sign.

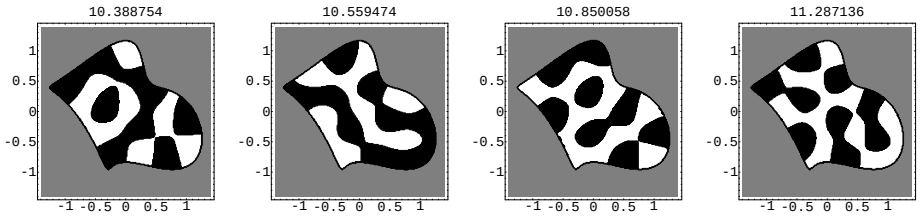

Figure 2: Plots of the 21th$\cdots$24th nodal domains associated to Ω_1

Simulation 2: Dirichlet/Neumann boundary conditions

To show the versatility of the MFS applied to the identification of eigenfrequencies and eigenmodes we will also present an example with a non simply connect domain $\Omega_E = \Omega_C \backslash \overline{B(0,1)}$, where Ω_C is the domain with boundary parameterized by

$$t \mapsto (3\cos(t), 2(\sin(t) + \cos(2t)) + 2).$$

In $\partial\Omega_C$ we impose a null Dirichlet boundary condition and in $\partial B(0,1)$ we impose a null Neumann boundary condition.

In this example we used 120 collocation and source points for the exterior boundary, and 60 for interior boundary. Note that in the case of a non simply connected domain, we must consider source points in all connected components of Ω^C.

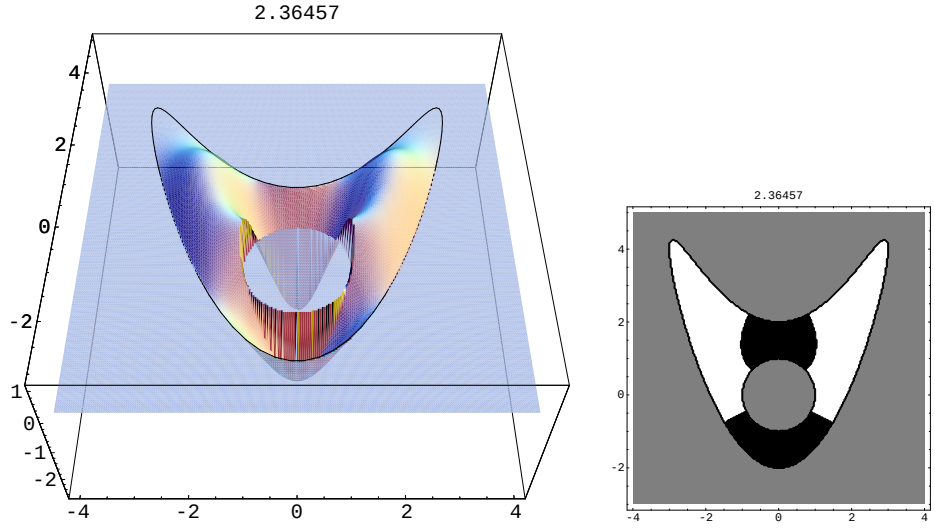

Figure 3: Eigenmode for the 4th eigenfrequency: plot and nodal domains.

Conclusions: In this brief account we presented the MFS method with an algorithm

for the choice of source points that has already been applied to the determination of eigenfrequencies and eigenmodes for hundreds of non trivial domains. The single example for a particular situation of mixed Dirichlet/Neumann illustrates the good results already obtained for other type of boundary conditions and non simply connected domains.

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