Advanced Meshless LBIEM for Thermoelastic Analysis of Nonhomogeneous Solids

J. Sladek[1], V. Sladek[1], Ch. Zhang[2]

Summary

A new computational method for solving transient thermoelastic boundary value problems in continuously nonhomogeneous solids, based on the meshless local Petrov-Galerkin (MLPG) method, is proposed in the present paper. If the test function in the MLPG is selected as the solution of the governing equation (fundamental solution) or as the solution of a part of the governing equations (parametrix), the unsymmetric weak form is transformed in such a way to the local boundary integral equations (LBIEs). The boundary-domain integral formulations for the temperature and the mechanical quantities with a static fundamental solution are derived in Laplace transform domain. The moving least squares (MLS) method is used for the approximation of physical quantities.

Introduction

The development of approximate methods for the numerical solutions of boundary value problems has attracted the attention of engineers, physicists and mathematicians for a long time. The finite element method (FEM), for modeling of complex problems in applied mechanics and related fields, is well established. It is a robust and thoroughly developed technique, but it is not without its own shortcomings. The boundary element method (BEM) has become an efficient and popular alternative to the FEM, especially for stress concentration problems, or for boundary value problems wherein a part of the boundary extends to infinity. In spite of the great success of the FEM and BEM as the most effective numerical tools for the solution of boundary value problems in many complex engineering applications, there has been a growing interest in the so-called meshless methods over the past decade [1,2].

In comparison to the conventional FEM or BEM, the meshless approach has certain advantages. In the meshless methods, nodal points are randomly spread over the domain of the analyzed body. Every node is surrounded by a simple surface centered at the collocation point. On the surface of subdomains the local boundary integral equations are written. In the present paper, nonstationary two-dimensional (2-d) thermoelastic problems in continuously nonhomogeneous medium are analyzed. It is assumed that the temperature and the displacement fields are uncoupled. Then, in the first step, the temperature field is analyzed. The transient heat conduction problem is described by a parabolic partial differential equation. To eliminate the time dependence of the

[1] Institute of Construction and Architecture, Slovak Academy of Sciences, 84503 Bratislava, Slovakia
[2] Department of Civil Engineering, University of Applied Sciences Zittau/Görlitz, D-02763 Zittau, Germany

differential equation, the Laplace transform technique is used. A pure boundary integral equation for the global boundary requires the knowledge of the fundamental solution in the Laplace transform domain, which is not known for a general continuously nonhomogeneous medium. In the present work, a simpler fundamental solution corresponding to the Poisson´s equation is adopted which leads to a boundary-domain integral formulation [3]. Since the boundary-domain formulation in the global BEM is numerically not efficient, we apply this formulation to small subdomains, which results in a local boundary integral equation formulation. If the thermal field is known, the mechanical quantities are obtained in the second step from the solution of the local boundary integral equations, which are reduced to the elastostatic LBIEs with known redefined body forces. Stationary thermoelastic problems in a homogeneous body have been analyzed by LBIEs in [4]. This paper is devoted to nonstationary thermoelastic analysis of continuously nonhomogeneous solids. The redefined body force is proportional to the temperature gradients.

Local boundary integral equations and their numerical solution

Consider a boundary value problem defined in the quasi-static uncoupled thermoelasticity for a continuously nonhomogeneous medium, which in 2-d is described by the governing equations

$$\theta_{,ii}(\mathbf{x},t) - \frac{1}{\kappa(\mathbf{x})}\frac{\partial\theta}{\partial t}(\mathbf{x},t) + \frac{k_{,i}}{k}(\mathbf{x})\theta_{,i}(\mathbf{x},t) = -\frac{1}{\kappa(\mathbf{x})}Q(\mathbf{x},t) \quad , \tag{1}$$

$$\mu(\mathbf{x})u_{i,kk}(\mathbf{x},t) + \frac{\mu(\mathbf{x})}{1-2\nu}u_{k,ki}(\mathbf{x},t) = -X_i(\mathbf{x},t) + \gamma(\mathbf{x})\theta_{,i}(\mathbf{x},t) -$$
$$-\mu_{,i}(\mathbf{x},t)\frac{2\nu}{1-2\nu}u_{k,k}(\mathbf{x},t) - \mu_{,j}(\mathbf{x})\left[u_{i,j}(\mathbf{x},t) + u_{j,i}(\mathbf{x},t)\right] \ , \tag{2}$$

where $\theta(\mathbf{x},t)$ and $u_i(\mathbf{x},t)$ are the temperature and the displacement fields, respectively, k and κ stand for the thermal conductivity and diffusivity, $Q(\mathbf{x},t)$ is the density of the body heat sources, μ and λ are Lame's constants, and $\gamma = (2\mu + 3\lambda)\alpha$, with α being the coefficient of the linear thermal expansion, respectively.

We assume an isotropic and linear elastic continuum with Young´s modulus depending on the Cartesian coordinates and Poisson´s ratio being constant. Since equations (1) and (2) are uncoupled, they can be solved separately. In the first step we solve the heat conduction equation (1). To eliminate the time dependence of temporal derivative from the diffusion equation (1) the Laplace transform is applied, which yields

$$\bar{\theta}_{,ii}(\mathbf{x},p) - \frac{p}{\kappa}\bar{\theta}(\mathbf{x},p) + \frac{k_{,i}}{k}(\mathbf{x})\bar{\theta}_{,i}(\mathbf{x},p) = -\frac{1}{\kappa}\bar{F}(\mathbf{x},p) \quad , \tag{3}$$

where p is the Laplace transform parameter and $\overline{F}(\mathbf{x},p)=\overline{Q}(\mathbf{x},p)+\theta(\mathbf{x},0)$ is the redefined body heat source in the Laplace transform domain with the initial boundary condition for the temperature field. The solution of the governing equation (3) can be found in a weak form by using the fundamental solution

$$\theta^*(\mathbf{x},\mathbf{y})=\frac{1}{2\pi}\ln\frac{1}{r} \quad . \tag{4}$$

In equation (4) $r=|\mathbf{x}-\mathbf{y}|$. The integral representation for the Laplace transform of the temperature can be derived for a subdomain Ω_s , which is a part of the analyzed domain Ω , i.e., $\Omega_s\subset\Omega$. The following local boundary integral equation (LBIE) holds over the subdomain Ω_s [3]

$$\overline{\theta}(\mathbf{y},p)=\int_{\partial\Omega_s}\frac{\partial\overline{\theta}}{\partial n}(\mathbf{x},p)\theta^*(\mathbf{x},\mathbf{y})d\Gamma-\int_{\partial\Omega_s}\overline{\theta}(\mathbf{x},p)\frac{\partial\theta^*}{\partial n}(\mathbf{x},\mathbf{y})d\Gamma-\int_{\Omega_s}\frac{p}{\kappa(\mathbf{x})}\overline{\theta}(\mathbf{x},p)\theta^*(\mathbf{x},\mathbf{y})d\Omega+$$

$$+\int_{\Omega_s}\frac{k_{,i}}{k}(\mathbf{x})\overline{\theta}_{,i}(\mathbf{x},p)\theta^*(\mathbf{x},\mathbf{y})d\Omega+\int_{\Omega_s}\frac{1}{\kappa(\mathbf{x})}\overline{F}(\mathbf{x},p)\theta^*(\mathbf{x},\mathbf{y})d\Omega \quad . \tag{5}$$

Note here that non of the boundary densities in the Laplace transformed domain are prescribed on the local boundary $\partial\Omega_s$ as long as it lies entirely inside Ω . To reduce the number of the unknowns on $\partial\Omega_s$ the concept of a companion solution can be utilized [4]. For a circular subdomain one can introduce a modified fundamental solution

$$\tilde{\theta}^*(\mathbf{x},\mathbf{y})=\frac{1}{2\pi}\ln\frac{r_0}{r} \quad , \tag{6}$$

which is vanishing on the boundary of the circular subdomain of radius r_0 . Since the integral equation is also valid for the modified fundamental solution $\tilde{\theta}^*(\mathbf{x},\mathbf{y})$, one can rewrite equation (5) as

$$\overline{\theta}(\mathbf{y},p)=-\int_{\partial\Omega_s}\overline{\theta}(\mathbf{x},p)\frac{\partial\tilde{\theta}^*}{\partial n}(\mathbf{x},\mathbf{y})d\Gamma-\int_{\Omega_s}\frac{p}{\kappa(\mathbf{x})}\overline{\theta}(\mathbf{x},p)\tilde{\theta}^*(\mathbf{x},\mathbf{y})d\Omega+$$

$$+\int_{\Omega_s}\frac{k_{,i}}{k}(\mathbf{x})\overline{\theta}_{,i}(\mathbf{x},p)\tilde{\theta}^*(\mathbf{x},\mathbf{y})d\Omega+\int_{\Omega_s}\frac{1}{\kappa(\mathbf{x})}\overline{F}(\mathbf{x},p)\tilde{\theta}^*(\mathbf{x},\mathbf{y})d\Omega \quad . \tag{7}$$

Once equation (7) has been solved numerically for discrete values of the Laplace transform parameter, the time-dependent values of the corresponding transformed quantities in the previous consideration can be obtained by an inverse transform. In the present analysis, the sophisticated Stehfest´s algorithm is used. In the second step, the Lame-Navier’s governing equation (2) is solved. The same procedure for deriving LBIEs, as in the case of the diffusion equation, is applied to obtain the LBIEs for the partial differential equations (2). As the test function in the weak formulation we use the displacement fundamental solution U_{ij} with Lame's constant $\mu = 1$. Then, the LBIEs on the subdomain Ω_s have the following form

$$u_j(\mathbf{y}) = \int_{\partial\Omega_s} \frac{1}{\mu(\mathbf{x})} t_i(\mathbf{x}) U_{ij}(\mathbf{x},\mathbf{y}) d\Gamma - \int_{\partial\Omega_s} T_{ij}(\mathbf{x},\mathbf{y}) u_i(\mathbf{x}) d\Gamma +$$

$$+ \int_{\Omega_s} \frac{1}{\mu(\mathbf{x})} \left[X_i(\mathbf{x}) + g_i(\mathbf{x}) - \gamma(\mathbf{x})\theta_{,i}(\mathbf{x}) \right] U_{ij}(\mathbf{x},\mathbf{y}) d\Omega \quad , \tag{8}$$

where $g_i(\mathbf{x}) = \mu_{,i}(\mathbf{x}) \frac{2\nu}{1-2\nu} u_{k,k}(\mathbf{x}) - \mu_{,j}(\mathbf{x}) \left[u_{i,j}(\mathbf{x}) + u_{j,i}(\mathbf{x}) \right]$.

To eliminate the tractions on $\partial\Omega_s$ the modified fundamental solution U_{ij}^* [5] is used. Since U_{ij}^* is vanishing on the boundary of the circular subdomain, we can rewrite equation (8) as

$$u_j(\mathbf{y}) = -\int_{\partial\Omega_s} T_{ij}^*(\mathbf{x},\mathbf{y}) u_i(\mathbf{x}) d\Gamma + \int_{\Omega_s} \frac{1}{\mu(\mathbf{x})} \left[X_i(\mathbf{x}) + g_i(\mathbf{x}) - \gamma(\mathbf{x})\theta_{,i}(\mathbf{x}) \right] U_{ij}^*(\mathbf{x},\mathbf{y}) d\Omega \quad . \tag{9}$$

The moving least squares (MLS) approximation is applied to interpolate the unknown/known densities in the LBIEs (7) and (9). Then, the LBIEs at the internal nodes are converted into a set of linear algebraic equations, which can be solved numerically.

Numerical results and discussions

Numerical results for a finite strip with a unidirectional variation of the thermal conductivity, diffusivity and expansion is presented to test the accuracy of the present LBIEM. An exponential spatial variation of the following form is considered

$$k(\mathbf{x}) = k_0 e^{\gamma x_1} \quad , \quad \kappa(\mathbf{x}) = \kappa_0 e^{\gamma x_1} \quad , \quad \alpha(\mathbf{x}) = \alpha_0 e^{-\delta x_1} \quad , \tag{10}$$

with $\kappa_0 = 0.17 \times 10^{-4} \; m^2 s^{-1}$ and $k_0 = 17 \; Wm^{-1} \deg^{-1}$. In particular, we have used the following exponential parameters $\gamma = \delta = 0.2 \; cm^{-1}$.

The Young's modulus is considered to be constant, since in many ceramic/metal functionally graded materials (FGM) systems on the zirconia/nickel basis the Young's modulus of both constituents are similar. However, the present method can be applied to more general cases with variable Young's modulus. On both opposite sides parallel to the x_2 -axis temperatures are prescribed. One side is kept to zero temperature, which is the initial value, and the other has the Heaviside step time variation $\theta(x_2,t) = T \cdot H(t)$ with $T = 1$ deg. On the top and bottom sides of the strip the heat flux vanishes. In our numerical calculations, a square with a side-length $a = 0.04$ m and a regular node distribution with equal 16 boundary and interior nodes are used for the MLS approximation. An excellent agreement between the numerical and the analytical results for the time variation of the temperature is observed in Fig. 1.

Figure 2 shows the results of the numerically computed stresses σ_{22} at $x_2 = a/2$ along the on x_1 -axis at two different time instants. The thermal stresses induced in the strip are given by

$$\sigma_{22}(x_1,t) = -\frac{\alpha(x_1)E}{1-\nu}\theta(x_1,t) \ . \tag{11}$$

Our numerical results are in a good agreement with the analytical ones (11). The maximal relative error for all computed nodes and over the whole analyzed time interval is less than 1%. In FGM strip the thermal expansion is decreasing with increasing x_1 - coordinate, see equation (10). Then, the stresses on the right-hand lateral side of the FGM strip with prescribed heat shock will be reduced in FGM strip in comparison with a homogeneous one.

References

1 Belytshko, T., Krongauz, Y., Organ, D., Fleming, M. and Krysl, P. (1996): “Meshles methods; An overview and recent developments”, *Computer Methods in Applied Mechanics and Engineering*, Vol. 139, pp. 3-47.

2 Atluri, S. N. and Shen, S.P. (2002): *The Meshless Local Petrov-Galerkin (MLPG) Method*, Tech Science Press.

3 Sladek, J., Sladek, V. and Zhang, Ch. (2003): “Transient heat conduction analysis in functionally graded materials by the meshless local boundary integral equation method”, *Computational Materials Science*, Vol. 28, pp. 494-504.

4 Sladek, J., Sladek, V. and Atluri, S.N. (2001): “A pure contour formulation for the meshless local boundary integral equation method in thermoelasticity”, *CMES: Computer Modeling in Engineering & Sciences*, Vol. 2, pp. 423-433.

5 Sladek, J., Sladek, V. and Zhang, Ch. (2003): “Application of meshless local Petrov-Galerkin (MLPG) method to elastodynamic problems in continuously nonhomogeneous solids”, *CMES: Computer Modeling in Engineering & Sciences*, Vol. 4, pp. 637-647.

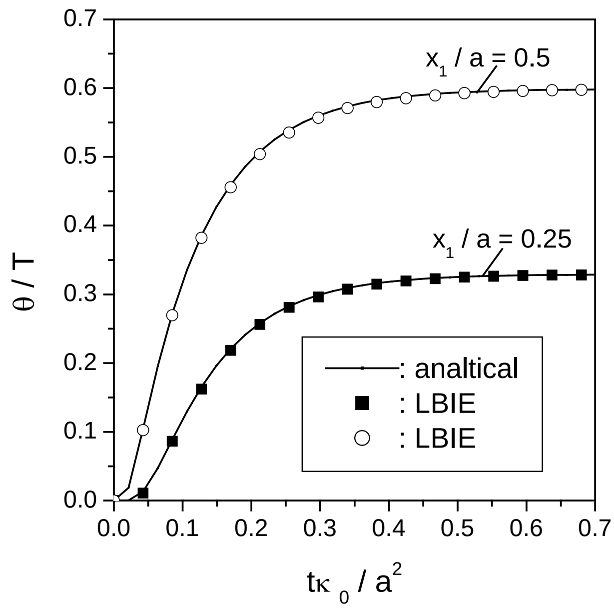

Fig. 1 Time variation of the temperature in a FGM strip (x_2 /a =0.5)

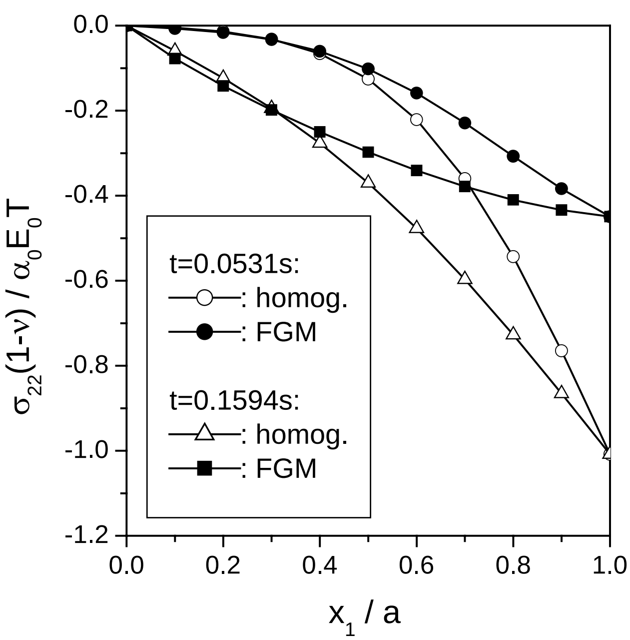

Fig. 2 Comparison of stress variation in homogeneous and FGM strip (x_2 /a=0.5)