Axisymmetric Bifurcation Analysis with Tangential Plasticity

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Summary

This article discusses about localized bifurcation modes corresponding to shear band formation and diffuse bifurcation modes of deformation, such as necking and bulging, for a cylindrical metallic specimen subjected to tensile or compressive loading under axisymmetric deformations. Also, conditions for the shear band inclination, the diffuse bifurcation, and the long and short wavelength bifurcation are discussed in relation to material properties and their state of stress. We employ the tangential-subloading surface model, in which tangential-plastic strain rate term is the necessary condition for formation of shear band and of diffuse bifurcation. Furthermore, their formation is severely affected by the normal-yield ratio describing the approach of magnitude of stress to that of the normal-yield state.

Introduction

The phenomenon of plastic instability is inevitably induced when a deformation becomes so very large and leads to bifurcation of deformation. Localized and diffuse bifurcations are widely observed in materials. Various theoretical analyses for localized and diffuse bifurcation modes of deformation (shear band, necking, bulging and buckling) have been attempted [1-4]. Results of these studies suggest the deficiency of traditional elastoplastic constitutive equations [5] in which the interior yield surface is a purely elastic domain, and the plastic strain rate is independent of the stress rate component tangential to the yield/loading surface. Several models have been proposed in which elastoplastic constitutive equations are extended to describe the tangential stress rate effect. The model proposed by Hashiguchi and Tsutsumi [5] is a model that applicable to an arbitrary loading process that includes unloading and reloading. It is formulated by introducing an additional strain rate, named tangential-plastic strain rate, induced by the deviatoric tangential stress rate into the subloading surface model [5]. This article explains how the existing bifurcation analyses of deformation in cylindrical specimen [1-4] can be extended to include the tangential-subloading surface model [5]. Moreover, the characteristic regimes are identified and the conditions for the shear band formation, the diffuse bifurcation, and the long and short wavelength bifurcation are discussed in relation to the state of stress and material parameters.

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Tangential-Subloading Surface Model

The normal yield and subloading surface are described as

$$f(\boldsymbol{\sigma}) = F(H), \quad f(\boldsymbol{\sigma}) = RF(H), \tag{1}$$

where the scalar H and R are the isotropic hardening variable and the normal-yield ratio, respectively.

Let it be assumed that the strain rate $\mathbf{D}$ is additively decomposed the elastic strain rate $\mathbf{D}^e$ and the inelastic strain rate $\mathbf{D}_i^p$ which is further decomposed into the normal-plastic strain rate $\mathbf{D}_n^p$ and the additional strain rate $\mathbf{D}_t^p$ [5], i.e.

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}_i^p, \quad \mathbf{D}_i^p = \mathbf{D}_n^p + \mathbf{D}_t^p, \tag{2}$$

where the elastic strain rate $\mathbf{D}^e$ and the normal-plastic strain rate $\mathbf{D}_n^p$ are given by

$$\mathbf{D}^e = \mathbf{E}^{-1}\mathring{\boldsymbol{\sigma}}, \quad \mathbf{D}_n^p = \lambda\,\mathbf{N}, \quad \mathbf{N} = \frac{\partial f}{\partial \boldsymbol{\sigma}} \Big/ \left\| \frac{\partial f}{\partial \boldsymbol{\sigma}} \right\| \quad (\|\mathbf{N}\| = 1), \tag{3}$$

$\boldsymbol{\sigma}$ is stress and, $(\circ)$ indicates the proper corotational rate with objectivity. The fourth-order tensor $\mathbf{E}$, the second order tensor $\mathbf{N}$ and λ are the elastic modulus, the normalized outward-normal of the subloading surface and the positive proportionality factor, respectively. $\mathbf{D}_t^p$ in Eq. (2) is called the tangential-plastic strain rate and induced by the stress rate component $\mathring{\boldsymbol{\sigma}}_t^*$ tangential to the subloading surface. $\mathring{\boldsymbol{\sigma}}_t^*$ called the tangential stress rate and is formulated as

$$\mathbf{D}_t^p = \frac{1}{M_t^p}\mathring{\boldsymbol{\sigma}}_t^*, \quad M_t^p = TR^{-b}. \tag{4}$$

M_t^p is a monotonically decreasing function of R, called the tangential-plastic modulus. $T\,(>0)$ and $b\,(\geq 1)$ are a material function and constant, respectively.

The elastoplastic constitutive equation for the present model is given as

$$\mathring{\boldsymbol{\sigma}} = \frac{M_t^p}{M_t^p + 2G}\left\{\mathbf{ED} - \frac{\mathrm{tr}(\mathbf{NED})}{M_n^p + \mathrm{tr}(\mathbf{NEN})}\left(\mathbf{EN} - 2G\frac{M_n^p}{M_t^p}\mathbf{N}\right) + \frac{2G}{3M_t^p}\mathrm{tr}(\mathbf{ED})\mathbf{I}\right\}. \tag{5}$$

where G and M_n^p are the shear moduli and the normal-plastic modulus, respectively. Adopt the simple elastoplastic material possessing the following von Mises yield condition with linear isotropic hardening [6].

$$f(\boldsymbol{\sigma}) = \sqrt{3/2}\,\|\boldsymbol{\sigma}^*\| = RF, \quad F = F_0 + c_i H, \quad \mathring{H} = \sqrt{2/3}\,\|\mathbf{D}_n^p\| \tag{6}$$

$$M_n^p = (2/3)c_i R + \sqrt{2/3}\,UF, \quad U = -u_R \ln R, \tag{7}$$

where c_i and u_R are the material constants and F_0 is the initial value of F.

Constitutive Relations

Consider the homogeneous deformation, the constitutive relationship, Eq. (5), can be expressed in the cylindrical coordinate system (see Fig. 1) as follows:

$$\left.\begin{array}{l}\mathring{\sigma}_{zz}-(\mathring{\sigma}_{rr}+\mathring{\sigma}_{\theta\theta})/2=2\mu^{*}(D_{zz}-(D_{rr}+D_{\theta\theta})/2)\,,\\ \mathring{\sigma}_{rr}-\mathring{\sigma}_{\theta\theta}=2\mu(D_{rr}-D_{\theta\theta})\,,\qquad \mathring{\sigma}_{rz}=2\mu D_{rz}\,,\end{array}\right\} \tag{8}$$

Noting that $\mathrm{tr}\mathbf{D}=0$ and $D_{r\theta}=D_{\theta z}=0$, instantaneous shear moduli μ^{*} and μ be given as follows:

$$\mu^{*}=G\frac{M_{n}^{p}}{M_{n}^{p}+2G}(\leq G)\,,\quad \mu=G\frac{M_{t}^{p}}{M_{t}^{p}+2G}(\leq G)\,. \tag{9}$$

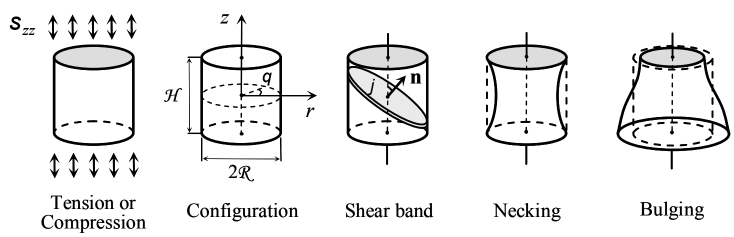

Fig. 1. The cylindrical specimen before and after deformation.

μ^{*} and μ in Eq. (9) are independent of the tangential-plastic and normal-plastic strain rate, respectively, and decrease monotonically with the increase in the value of the normal-yield ratio R, as shown in Fig. 2.

The equilibrium equation can be written as

$$\mathrm{div}\,\dot{\boldsymbol{\Pi}}=0,\quad \dot{\boldsymbol{\Pi}}=\mathring{\boldsymbol{\sigma}}+(\mathrm{tr}\mathbf{D})\boldsymbol{\sigma}-\boldsymbol{\sigma}\mathbf{D}+\mathbf{W}\boldsymbol{\sigma}\,,\quad \mathring{\boldsymbol{\sigma}}=\dot{\boldsymbol{\sigma}}+\boldsymbol{\sigma}\mathbf{W}-\mathbf{W}\boldsymbol{\sigma}\,. \tag{10}$$

where $\dot{\boldsymbol{\Pi}}$, $\mathring{\boldsymbol{\sigma}}$ and $\mathbf{W}$ are the nominal stress rate, Jaumann stress rate and spin tensor, respectively.

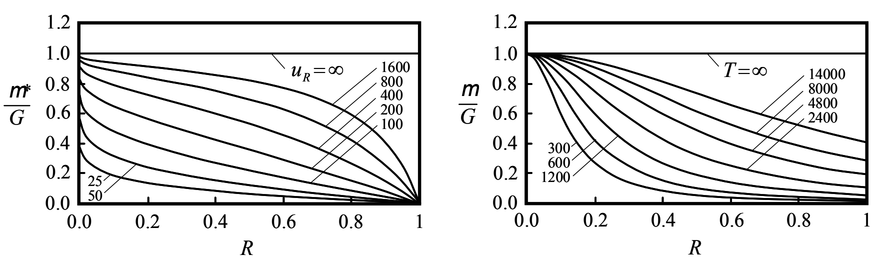

Fig. 2. Instantaneous shear moduli μ^{*} and μ vs. the normal-yield ratio R, ($c_i = 150$).

Localized Bifurcation

Two conditions must be satisfied for the shear band formation [7], which can be expressed as follows (Fig. 1):

$$v_{i,j} = g_i n_j, \quad \dot{\Pi}_{ij} n_j = 0, \qquad (i, j = r, z). \tag{11}$$

where $\mathbf{g}$ is the jump vector of the velocity gradient and $\mathbf{n}$ is the unit vector normal to the shear band. Substituting Eq. (10) into (11) yields:

$$(\mu-\sigma)n_r^4 + (3\mu^* - \mu)n_r^2 n_z^2 + (\mu+\sigma)n_z^4 = 0, \quad \sigma = \sigma_{zz}/2. \tag{12}$$

The solutions of n_r/n_z in Eq. (12) are classified as elliptic complex (EC), elliptic imaginary (EI), parabolic (P) and hyperbolic (H) regimes. The inclination angle φ of the shear band for the EC-H boundary can be obtained from Eq. (12) as

$$\varphi \equiv \tan^{-1}\left(\frac{n_r}{n_z}\right) = \tan^{-1}\sqrt{-\frac{3\mu^* - \mu}{2(\mu-\sigma)}}. \tag{13}$$

Diffuse Bifurcation Modes

The equilibrium equation (10) in homogeneous deformation for axisymmetric conditions reduces to

$$\frac{\partial \dot{\Pi}_{rr}}{\partial r} + \frac{\partial \dot{\Pi}_{rz}}{\partial z} + \frac{1}{r}(\dot{\Pi}_{rr} - \dot{\Pi}_{\theta\theta}) = 0, \quad \frac{\partial \dot{\Pi}_{zz}}{\partial z} + \frac{\partial \dot{\Pi}_{zr}}{\partial r} + \frac{1}{r}\dot{\Pi}_{zr} = 0. \tag{14}$$

Introducing the stream function Ψ such as $v_r = \partial\Psi/\partial z$ and $v_z = -\partial(r\Psi)/r\partial r$, Eq. (14) leads to:

$$(\mu-\sigma)O_r^2 + (3\mu^* - \mu)\frac{\partial^2}{\partial^2 z}O_r + (\mu+\sigma)\frac{\partial^4\Psi}{\partial^4 z} = 0, \quad O_r(\Psi) = \frac{\partial}{\partial r}\left(\frac{\partial(r\Psi)}{r\partial r}\right). \tag{15}$$

Ψ in Eq. (15), which gives rise to the diffuse bifurcation modes is given by

$$\Psi(r,z) = \psi(r)\cos(\zeta z), \quad \zeta = m\pi/\mathcal{H}, \quad m = 1, 2, 3, \ldots \tag{16}$$

Substituting Eq. (16) into Eq. (15), leads to

$$(O_r + \zeta^2\iota_1^2)(O_r + \zeta^2\iota_2^2)\psi(r) = 0. \tag{17}$$

where ι_1 and ι_2 are the roots of the following characteristic equation.

$$(\mu-\sigma)\iota^4 + (3\mu^* - \mu)\iota^2 + (\mu+\sigma) = 0. \tag{18}$$

With boundary conditions; $\dot{\Pi}_{rr} = 0$ and $\dot{\Pi}_{zr} = 0$ on $r = \mathcal{R}$, Eq.(14) becomes:

$$\left.\begin{aligned} &O_r(y) + z^2 y = 0, \\ &\frac{1}{r}(m-s)\frac{d}{dr}(rO_r(y)) - z^2\Big((3m^*-s)\frac{dy}{dr} + \frac{1}{r}(3m^* - 2m - s)y\Big) = 0. \end{aligned}\right\} \tag{19}$$

The roots of Eq. (18) are classified as the elliptic complex (EC), elliptic imaginary (EI), hyperbolic (H) and parabolic (P) regimes. Further, the solution of Eq. (17) in the elliptic complex regime has the form

$$y(r) = M J_1(zj\, r) + \bar{M} J_1(z\bar{j}\, r), \tag{20}$$

M and $\bar{M}$ are constants and $J_1(\ )$ is Bessel function of the first kind of order one. $j = p + iq$ and $\bar{j} = p - iq$ are the roots of Eq. (18). Substituting Eq. (20) into boundary conditions (19) leads to the following eigenvalue equation:

$$\frac{(1-j^2)J_1(wj)}{(1-\bar{j}^2)J_1(w\bar{j})} = \frac{wj((m-s)j^2 + (3m^*-s))J_0(wj) - 2mJ_1(wj)}{w\bar{j}((m-s)\bar{j}^2 + (3m^*-s))J_0(w\bar{j}) - 2mJ_1(w\bar{j})}, \tag{21}$$

$w = z\mathcal{R} = mp\,\mathcal{R}/\mathcal{H}$ is the wavelength. p and q satisfy the following equations:

$$p^2 + q^2 = \Big(\frac{m+s}{m-s}\Big)^{1/2}, \quad p^2 - q^2 = -\frac{3m^* - m}{2(m-s)} \tag{22}$$

Eigenvalue equation (21) for the long wavelength limit $(w \to 0)$ becomes: $s/3m^* = 0.5$ and for the short wavelength limit $(w \to \infty)$:

$$\frac{s}{3m^*} = \frac{1}{2} + \frac{1}{2}\frac{m}{3m^*} + \frac{s}{3m^*}\Big(\frac{m-s}{m+s}\Big)^{1/2}. \tag{23}$$

Fig. 3 represents the bifurcation regimes as a function of the dimensionless variables $s/3m^*$ and $m/3m^*$. The $(s/3m^*, m/3m^*)$ trajectories for the long wavelength limit, short wavelength limit and typical wavelength w ranging from 1.0 to 2.0 are also depicted in Fig. 3. Fig. 4 represents the lowest bifurcation stress as a function of the wavelength w in several values of $3m^*/m$.

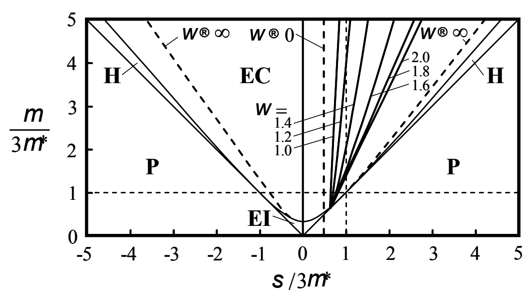

Fig. 3. Characteristic regimes and (♦/3❍, ❍3❍) trajectories.

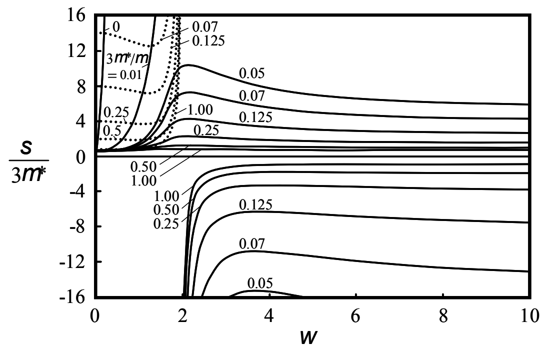

Fig. 4. Lowest bifurcation stresses with the variation of 3❍/❍

Concluding Remarks

The localized and diffuse bifurcation of deformation can simulated reasonably well by the tangential-subloading surface mode, and results reveal that pre-peak bifurcation of diffuse modes is always possible and can occur preceding the formation of strain localization. Further, the diffuse bifurcations may result in or act a trigger for the premature localization of deformation.

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