The Novel Concept "Tangential Relaxation"

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Summary

The novel concept "*tangential stress rate relaxation*", abbreviated as "*tangential relaxation*", is proposed in order to predict rigorously the plastic instability phenomena in which the stress rate has a tangential component deviating severely from the proportional loading. Further, the constitutive equation based on this concept is formulated.

Introduction

Traditional plasticity is concerned only with the stress rate component normal to the yield surface but is independent of the tangential component. Thus, it predicts an unrealistically stiff mechanical response in plastic instability phenomena in which the stress rate has a tangential component deviating severely from the proportional loading. In order to improve this defect in the traditional theory, various constitutive models have been proposed so far. Among them only the tangential inelasticity model [1] which incorporates the inelastic strain rate induced by the stress rate component tangential to the subloading surface is applicable to the general loading process, which is regarded as the modification of Rudnicki and Rice's [2] rate form of the J_2-deformation theory by the concept of the subloading surface model [3, 4]. However, it is not derived from the physically rigorous background.

In this article the novel concept "*tangential stress rate relaxation*", abbreviated as "*tangential relaxation*", is proposed in order to predict rigorously the plastic instability phenomena and the constitutive equation based on this concept is formulated.

Outline of the Subloading Surface Model

Let the strain rate $\mathbf{D}$ be additively decomposed into the elastic strain rate $\mathbf{D}^e$ and the inelastic strain rate $\mathbf{D}^p$, i.e.

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p, \tag{1}$$

where $\mathbf{D}^e$ is given by

$$\mathbf{D}^e = \mathbf{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}}. \tag{2}$$

$\boldsymbol{\sigma}$ is the Cauchy stress and $(\circ)$ indicates the proper corotational rate and the fourth-order tensor $\mathbf{E}$ is the elastic modulus.

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The following *subloading surface* is introduced.

$$f(\hat{\boldsymbol{\sigma}}, \mathbf{H}) = RF(H), \tag{3}$$

where

$$\hat{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \boldsymbol{\alpha} \cdot \tag{4}$$

The scalar H and the second-order tensor $\mathbf{H}$ are the isotropic and the anisotropic hardening variables, respectively, $\boldsymbol{\alpha}$ is the kinematic hardening variable, i.e. the back stress. The function f is assumed to be homogeneous of degree one in the stress $\hat{\boldsymbol{\sigma}}$. R is the ratio of the size of the subloading surface to that of the normal-yield surface and is called the *normal-yield ratio*. Its evolution equation is given as follows:

$$\overset{\bullet}{R} = U(R)\|\mathbf{D}^p\| \ \text{ for } \mathbf{D}^p \neq \mathbf{0}, \tag{5}$$

where U is a monotonically-decreasing function of the normal-yield ratio R, fulfilling the conditions

$$U(R) = \begin{cases} \infty & \text{for } R = 0, \\ 0 & \text{for } R = 1, \end{cases} \tag{6}$$
$$(U(R) < 0 \quad \text{for } R > 1).$$

Let the function U satisfying Eq. (6) be simply given by

$$U(R) = -u \ln R, \tag{7}$$

where u is a material constant.

Further, introduce the associated flow rule

$$\mathbf{D}^p = \lambda \mathbf{N}, \tag{8}$$

where λ is a positive proportionality factor and

$$\mathbf{N} \equiv \frac{\partial f(\hat{\boldsymbol{\sigma}}, \mathbf{H})}{\partial \boldsymbol{\sigma}} \Bigg/ \left\| \frac{\partial f(\hat{\boldsymbol{\sigma}}, \mathbf{H})}{\partial \boldsymbol{\sigma}} \right\| \quad (\|\mathbf{N}\| = 1) \cdot \tag{9}$$

The proportionality factor λ is obtained by substituting Eqs. (5) and (8) into the time-diffrentiation of Eq. (3) leads to

$$\lambda = \frac{\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}})}{M^p}, \tag{10}$$

where

$$M^p \equiv \mathrm{tr}\Big[\mathbf{N}\Big[\Big(\frac{F'}{F}h + \frac{U}{R}\Big)\hat{\boldsymbol{\sigma}} + \mathbf{a} - \frac{1}{RF}\mathrm{tr}\Big\{\frac{\partial f(\hat{\boldsymbol{\sigma}}, \mathbf{H})}{\partial \mathbf{H}}\mathbf{h}\Big\}\hat{\boldsymbol{\sigma}}\Big]\Big]. \tag{11}$$

$$F' \equiv \frac{dF}{dH}, \ h \equiv \frac{\overset{\bullet}{H}}{\lambda}, \ \mathbf{h} \equiv \frac{\overset{\circ}{\mathbf{H}}}{\lambda}, \ \mathbf{a} \equiv \frac{\overset{\circ}{\boldsymbol{\alpha}}}{\lambda}. \tag{12}$$

The strain rate is given from Eqs. (1), (2), (8) and (10) as

$$\mathbf{D} = \mathbf{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}} + \frac{1}{M^p}\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}})\mathbf{N}. \tag{13}$$

Tangential Relaxation

It can be written from Eq. (13) that

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{E}\mathbf{D} - \frac{1}{M^p}\mathbf{E}\overset{\circ}{\boldsymbol{\sigma}}_N, \tag{14}$$

where $\overset{\circ}{\boldsymbol{\sigma}}_N$ is the normal-stress rate, i.e.

$$\overset{\circ}{\boldsymbol{\sigma}}_N \equiv \mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}})\mathbf{N} = (\mathbf{N}\otimes\mathbf{N})\overset{\circ}{\boldsymbol{\sigma}}. \tag{15}$$

It is observed in Eq. (14) that the relaxation relevant to the normal-stress rate is induced. Let it be called the "*normal relaxation*". Now, it is postulated that the relaxation relevant to the deviatoric-tangential stess rate is also induced. Let it be referred to as the "*deviatoric-tangential relaxation*", abbreviated as "*tangential relaxation*". In what follows, let the extended subloading surface model, called the *tangential-subloading surface model*, be formulated.

Now, let Eq. (14) be extended as

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{E}\Big(\mathbf{D} - \frac{1}{M^p}\overset{\circ}{\boldsymbol{\sigma}}_N - \frac{1}{M^t}\overset{\circ}{\boldsymbol{\sigma}}{}^*_T\Big), \tag{16}$$

where $\overset{\circ}{\boldsymbol{\sigma}}{}^*_T$ is called the *deviatoric-tangential relaxation stress rate* and is given as

$$\overset{\circ}{\boldsymbol{\sigma}}{}^*_T \equiv \Big(\frac{\overset{\circ}{\boldsymbol{\sigma}}{}^*_t}{\|\overset{\circ}{\boldsymbol{\sigma}}{}^*_t\|} + d_n\mathbf{n}^*\Big)\|\overset{\circ}{\boldsymbol{\sigma}}{}^*_t\| \quad \text{or} \quad \overset{\circ}{\boldsymbol{\sigma}}{}^*_T \equiv \overset{\circ}{\boldsymbol{\sigma}}{}^*_t + d_n\mathbf{n}^*\|\overset{\circ}{\boldsymbol{\sigma}}{}^*_t\|. \tag{17}$$

The deviatoric-tangential stress rate $\overset{\circ}{\boldsymbol{\sigma}}{}^*_t$ is given as follows:

$$\overset{\circ}{\boldsymbol{\sigma}}{}^* = \overset{\circ}{\boldsymbol{\sigma}}{}^*_n + \overset{\circ}{\boldsymbol{\sigma}}{}^*_t, \tag{18}$$

where the deviatoric-normal and tangential components of arbitrary second-order tensor $\mathbf{A}$ is defined as

$$\left.\begin{array}{l} \mathbf{A}^*_n \equiv \mathrm{tr}(\mathbf{n}^*\mathbf{A})\mathbf{n}^* = (\mathbf{n}^*\otimes\mathbf{n}^*)\mathbf{A}, \\ \mathbf{A}^*_t \equiv \mathbf{A}^* - \mathbf{A}^*_n = \mathbf{A}^* - \mathrm{tr}(\mathbf{n}^*\mathbf{A})\mathbf{n}^* = (\breve{\mathbf{I}}^* - \mathbf{n}^*\otimes\mathbf{n}^*)\mathbf{A} \end{array}\right\} \tag{19}$$

with

$$\mathbf{n}^* \equiv \Big(\frac{\partial f(\hat{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\Big)^* \Big/ \Big\|\Big(\frac{\partial f(\hat{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\Big)^*\Big\| = \frac{\mathbf{N}^*}{\|\mathbf{N}^*\|} \quad (\|\mathbf{n}^*\| = 1). \tag{20}$$

$(\ )^*$ stands for the deviatoric component and $\breve{\mathbf{I}}^*$ is the fourth-order *deviatoric transformation tensor*, i.e.

$$\breve{I}^*_{ijkl} \equiv \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl}\,. \tag{21}$$

The material function M^t, called the *tangential-relaxation modulus*, is a monotonically decreasing function of R and is simply given by

$$M^t = \frac{1}{\xi R^n}, \tag{22}$$

where n is a material constant and ξ is a material parameter which is a function of stress and plastic internal variables in general: a material constant for metals and a function of stress for frictional materials. d_n is a material constant by which the relaxation is induced in the inward-normal direction to the subloading surface.

The strain rate is expressed in terms of the stress rate from Eqs. (15), (16) and (17) as

$$\mathbf{D} = \mathbf{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}} + \frac{1}{M^p}\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}})\mathbf{N} + \frac{1}{M^t}(\overset{\circ}{\boldsymbol{\sigma}}{}^*_t + d_n\|\overset{\circ}{\boldsymbol{\sigma}}{}^*_t\|\mathbf{n}^*)\,. \tag{23}$$

Then, the strain rate is additively decomposed into the elastic strain rate $\mathbf{D}^e$ and the inelastic strain rate $\mathbf{D}^i$, while the latter is further additively decomposed into the plastic strain rate $\mathbf{D}^p$ and the tangential strain rate $\mathbf{D}^t$, i.e.

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^i, \quad \mathbf{D}^i = \mathbf{D}^p + \mathbf{D}^t, \tag{24}$$

while the tangential strain rate is given for Eq. (23) as follows:

$$\mathbf{D}^t = \frac{1}{M^t}(\overset{\circ}{\boldsymbol{\sigma}}{}^*_t + d_n\|\overset{\circ}{\boldsymbol{\sigma}}{}^*_t\|\mathbf{n}^*)\,. \tag{25}$$

The positive proportionality factor in the associated flow rule (8) is expressed in terms of strain rate with the tangential stress rate, rewriting λ by Λ, from Eq. (23) as follows:

$$\Lambda = \frac{\mathrm{tr}(\mathbf{NED}) - \frac{1}{M^t}\mathrm{tr}\{\mathbf{NE}(\overset{\circ}{\boldsymbol{\sigma}}{}^*_t + d_n\mathbf{n}^*\|\overset{\circ}{\boldsymbol{\sigma}}{}^*_t\|)\}}{M^p + \mathrm{tr}(\mathbf{NEN})} \left(= \frac{\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}})}{M^p}\right). \tag{26}$$

The loading criterion for the plastic strain rate is given from Eq. (26) as follows [6]:

$$\left.\begin{array}{l} \mathbf{D}^p \neq \mathbf{0}:\ \mathrm{tr}(\mathbf{NED}) - \frac{1}{M^t}\mathrm{tr}\{\mathbf{NE}(\overset{\circ}{\boldsymbol{\sigma}}{}^*_t + d_n\mathbf{n}^*\|\overset{\circ}{\boldsymbol{\sigma}}{}^*_t\|)\} > 0, \\ \mathbf{D}^p = \mathbf{0}:\ \text{otherwise} \end{array}\right\} \tag{27}$$

since it can be assumed that $M^p + \mathrm{tr}(\mathbf{NEN}) > 0$, while the tangential strain rate $\mathbf{D}^t$ is always induced for $\overset{\circ}{\boldsymbol{\sigma}}{}_t^* \neq \mathbf{0}$. Eq. (23) is rate-nonlinear and thus an inverse expression becomes rather complicated form. It should be noted that the loading criterion has to be defined essentially by the sign of the proportionality factor Λ as has been revealed by Hashiguchi [5], while it has been defined merely by the quantity $\mathrm{tr}(\mathbf{NED})$ in the traditional elastoplastic constitutive equation and even in the past tangential-subloading surface model [1, 6, 7].

Hereafter, assume that the elastic modulus tensor $\mathbf{E}$ is given by Hooke's type, i.e.

$$E_{ijkl} = \left(K - \tfrac{2}{3}G\right)\delta_{ij}\delta_{kl} + G(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{28}$$

where K and G are the elastic bulk modulus and the elastic shear modulus, respectively, which leads to the relations $\mathrm{tr}(\mathbf{SET}^*) = 2G\mathrm{tr}(\mathbf{ST}^*)$ and $\breve{\mathbf{I}}^*\mathbf{ET} = \mathbf{ET}^* = 2G\mathbf{T}^*$ for arbitrary second-order tensors $\mathbf{S}$ and $\mathbf{T}$. Then, Eq. (16) with Eqs. (15) and (17) is rewritten as

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{ED} - \frac{\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}})}{M^p}\mathbf{EN} - \frac{2G}{M^t}(\overset{\circ}{\boldsymbol{\sigma}}{}_t^* + d_n\mathbf{n}^*\|\overset{\circ}{\boldsymbol{\sigma}}{}_t^*\|), \tag{29}$$

from which one has

$$\left.\begin{aligned}
&\overset{\circ}{\boldsymbol{\sigma}}{}^* = 2G\left\{\mathbf{D}^* - \frac{\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}})}{M^p}\mathbf{N}^* - \frac{1}{M^t}(\overset{\circ}{\boldsymbol{\sigma}}{}_t^* + d_n\|\overset{\circ}{\boldsymbol{\sigma}}{}_t^*\|\mathbf{n}^*)\right\},\\
&\mathrm{tr}(\mathbf{n}^*\overset{\circ}{\boldsymbol{\sigma}}{}^*)\mathbf{n}^* = 2G\left\{\mathrm{tr}(\mathbf{n}^*\mathbf{D}^*) - \frac{\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}})}{M^p}\mathrm{tr}(\mathbf{n}^*\mathbf{N}^*) - \frac{d_n}{M^t}\|\overset{\circ}{\boldsymbol{\sigma}}{}_t^*\|\right\}\mathbf{n}^*
\end{aligned}\right\} \tag{30}$$

Noting $\mathbf{N}_t^*\ (= \mathbf{N}^* - \mathrm{tr}(\mathbf{n}^*\mathbf{N}^*)\}\mathbf{n}^*) = 0$, it is obtained from Eq. (30) that

$$\overset{\circ}{\boldsymbol{\sigma}}{}_t^* = \frac{2GM^t}{M^t + 2G}\mathbf{D}_t^*, \tag{31}$$

where note that the deviatoric tangential-stress rate $\overset{\circ}{\boldsymbol{\sigma}}{}_t^*$ is simply proportional to the deviatoric tangential-strain rate $\mathbf{D}_t^*\ (= \mathbf{D}^* - \mathrm{tr}(\mathbf{n}^*\mathbf{D}^*)\mathbf{n}^*)$.

Substituting Eq. (31) and noting

$$\mathrm{tr}(\mathbf{NE}\overset{\circ}{\boldsymbol{\sigma}}{}_t^*)\ (= 2G\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}}{}_t^*)) = 0, \tag{32}$$

the proportionality factor Λ in Eq. (26) can be described in terms of strain rate as

$$\Lambda = \frac{\mathrm{tr}(\mathbf{NED}) - d_n \dfrac{2G}{M^t + 2G} \mathrm{tr}(\mathbf{Nn}*) \left\| \mathbf{D}_t^* \right\|}{M^p + \mathrm{tr}(\mathbf{NEN})} . \tag{33}$$

The inverse expression, i.e. the analytical expression of stress rate in terms of strain rate is derived as follows:

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{ED} - \frac{\mathrm{tr}(\mathbf{NED}) - d_n \dfrac{2G}{M^t + 2G} \mathrm{tr}(\mathbf{Nn}*) \left\| \mathbf{D}_t^* \right\|}{M^p + \mathrm{tr}(\mathbf{NEN})} \mathbf{EN} - \frac{(2G)^2}{M^t + 2G} (\mathbf{D}_t^* + d_n \left\| \mathbf{D}_t^* \right\| \mathbf{n}*) . \tag{34}$$

The loading criterion is given as

$$\left. \begin{array}{l} \mathbf{D}^p \neq \mathbf{0} : \mathrm{tr}(\mathbf{NED}) - d_n \dfrac{2G}{M^t + 2G} \mathrm{tr}(\mathbf{Nn}*) \left\| \mathbf{D}_t^* \right\| > 0, \\ \mathbf{D}^p = \mathbf{0} : \text{otherwise} \end{array} \right\} \tag{35}$$

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