Reduced Order Implicit Integration Algorithms for Frictional Materials

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Summary

In this contribution the two-step plasticity algorithm proposed by Etse et al [1] and Macari et al [2] has been modified to allow for update of the hardening/softening parameter within the same iteration loop as the effective mean normal p and the deviator stress q invariants. Two different algorithms are presented. In the first one, the correction for the Lode angle is applied by means of a two-step procedure. In the second algorithm, all unknown quantities are updated within the same iteration loop. Examples of application of the proposed algorithms to a material whose plastic response is governed by the so-called Desai surface [3] are described.

Introduction

In computational plasticity, the increments of plastic strains and of the hardening parameter(s) are determined by integration of the flow rule(s) and the hardening law(s) over a time step Δt. Because of their improved stability, implicit methods of time integration have become very popular. Typically a potential function $f(\sigma, \alpha)$ is utilized in the definition of the flow rule $\dot{\varepsilon}_p = \dot{\lambda}(\partial f / \partial \sigma)$. If the variables of the potential function can not be separated (i.e. $f(\sigma, \alpha) \neq g(\sigma) + h(\alpha)$) then, after yield, a set of nonlinear equations can be set up consisting of the six strain compatibility equations, an equation expressing the change in the hardening parameter and the consistency equation

$$
\begin{aligned}
& D^{-1}\left({}^{n+1}\sigma - {}^{n+1}_{\ e}\sigma\right) + {}^{n+1}\Delta\varepsilon_p = 0 \\
& {}^{n+1}\Delta\alpha = \Delta\alpha\left({}^{n+1}\sigma, {}^{n+1}\Delta\varepsilon_p\right) \\
& {}^{n+1}f\left({}^{n+1}\sigma, {}^{n+1}\varepsilon_p\right) = 0
\end{aligned}
\tag{1}
$$

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These represent a system of 8 nonlinear equations, which are typically solved via Newton-Raphson iterations. Because this set of equations must be solved at every integration point of a finite element the process is time consuming. Over the years, attempts have been made to reduce the order of the system. For surfaces, which are only functions of the effective mean normal stress p and the deviator stress q, Aravas [5] has proposed a very efficient algorithm, which reduces the order of the nonlinear system to two. For surfaces which in addition to p and q are also functions of the Lode angle, Etse et al [1] and Macari et al [2] have proposed an algorithm in which a two step procedure is utilized for reduction of the number of unknowns.

Algorithmic aspects

The original algorithm of Etse et al [1] and Macari et al [2] is based on the Closest Point Projection Method (CPPM), in which stress reduction to the yield surface, can be done by minimizing the complementary energy equation:

$$E_A\left({}^{n+1}\sigma\right)=\frac{1}{2}\left({}^{n+1}_{e}\sigma-{}^{n+1}\sigma\right):{}^{n+1}\left(D^e\right)^{-1}:\left({}^{n+1}_{e}\sigma-{}^{n+1}\sigma\right) \tag{2}$$

in which ${}^{n+1}_{e}\sigma$ is the trial elastic stress, ${}^{n+1}\sigma$ is the plastically admissible stress and D^e is the elastic tangent stiffness tensor.

In case of isotropic yield criteria with an isotropic hardening/softening rule, the energy $E_A\left({}^{n+1}\sigma\right)$ can be reformulated in stress-invariant space as:

$$\begin{aligned}2E_A=&\frac{1}{K}\left({}^{n+1}_{e}p-{}^{n+1}p\right)^2+\frac{1}{3G}\left({}^{n+1}_{e}q\cos\left({}^{n+1}_{e}\vartheta\right)-{}^{n+1}q\cos\left({}^{n+1}\vartheta\right)\right)^2+\\&\frac{1}{3G}\left({}^{n+1}_{e}q\sin\left({}^{n+1}_{e}\vartheta\right)-{}^{n+1}q\sin\left({}^{n+1}\vartheta\right)\right)^2\end{aligned} \tag{3}$$

in which p is the effective mean normal stress, q is the deviator stress and ϑ the Lode angle:

$$p=\frac{1}{3}I_1\,,\quad q=\sqrt{3J_2}\quad\text{and}\quad \vartheta=\frac{1}{3}\arccos\left(\frac{3\sqrt{3}}{2}\frac{J_3}{J_2\sqrt{J_2}}\right) \tag{4}$$

I_1, J_2 and J_3 are the stress invariants.

To comply with the plasticity consistency constraint ($F({}^{n+1}p, {}^{n+1}q, {}^{n+1}\vartheta, {}^{n+1}\alpha) \leq 0$), the following expression can be minimized

$$L = E_A\left({}^{n+1}p, {}^{n+1}q, {}^{n+1}\vartheta\right) - {}^{n+1}\lambda\, F({}^{n+1}p, {}^{n+1}q, {}^{n+1}\vartheta, {}^{n+1}\alpha) \qquad (5)$$

To minimize L, its derivatives with respect to the invariants and the plasticity consistency parameter λ should go to zero. This leads to the following set of non-linear equations:

$$ {}^{n+1}p - {}^{n+1}_{e}p + {}^{n+1}\mu\left(\frac{K}{3G}\right)\frac{\partial F}{\partial p} = 0 \qquad (6)$$

$$ {}^{n+1}q - {}^{n+1}_{e}q\cos\left({}^{n+1}_{e}\vartheta - {}^{n+1}\vartheta\right) + {}^{n+1}\mu\frac{\partial F}{\partial q} = 0 \qquad (7)$$

$$F({}^{n+1}p, {}^{n+1}q, {}^{n+1}\vartheta, {}^{n+1}\alpha) = 0 \qquad (8)$$

$$ {}^{n+1}q\, {}^{n+1}_{e}q\sin\left({}^{n+1}_{e}\vartheta - {}^{n+1}\vartheta\right) - {}^{n+1}\mu\frac{\partial F}{\partial \vartheta} = 0 \qquad (9)$$

with ${}^{n+1}\mu = 3G\, {}^{n+1}\lambda$. Solving this system of equations is done in a two level iterative procedure. On the basis of the new elastic stress state ${}^{n+1}_{e}\sigma$ estimates for the stress invariants are calculated: ${}^{n+1}_{e}p$, ${}^{n+1}_{e}q$ and ${}^{n+1}_{e}\vartheta$.

Using a Newton-Raphson iteration scheme Eqs. (6) to (8) are solved for ${}^{n+1}p$, ${}^{n+1}q$ and ${}^{n+1}\alpha$ (the parameter λ or μ can be expressed as a function of the hardening/softening parameter α, which will be shown later). After substituting these new parameter values in Eq. (9), ${}^{n+1}\vartheta$ can be determined, again by the Newton-Raphson method. This new value of ${}^{n+1}\vartheta$ causes the stress-state ${}^{n+1}\sigma$ to change, so new values for ${}^{n+1}p$, ${}^{n+1}q$ and ${}^{n+1}\alpha$ have to be found, and so on. This iteration process ends when the difference between two consecutive values of ${}^{n+1}\vartheta$ is small enough.

Including the hardening parameter into the first phase is different from the approach described in Macari et al. [2]. In their work the hardening/softening parameter is calculated by an iterative loop around a two level iterative procedure. According to the authors, this is necessary if the hardening/softening parameter is of a highly non-linear nature. As it is shown in the next section, the hardening function used in this contribution, is non-linear but sufficiently smooth to be solved in the first stage of the two level iterative procedure. Also solving the three invariants together with the hardening/softening parameter has been implemented.

Desai yield surface

The Desai yield surface [3] is utilized in this contribution as the flow function:

$$F = \frac{q^2}{3p_a^2} - \left(-\alpha \left(\frac{3p+R}{p_a} \right)^n + \gamma \left(\frac{3p+R}{p_a} \right)^2 \right) \left(1 - \beta \cos(3\vartheta)\right)^m \tag{10}$$

Parameter γ is related to the ultimate strength of the material, R represents the tri-axial strength in tension and p_a is the atmospheric pressure. Parameter n is related to the state of stress at which the material response changes from compaction to dilatation. Finally parameter m is a constant, which has the value –0.5 for many geological materials [4]. Parameter α describes the isotropic hardening/softening. In the context of this contribution it has been defined as:

$$\alpha = \alpha_0 \exp\left(\kappa_c \xi\right) \quad \leftrightarrow \quad \xi = \frac{1}{\kappa_c} \ln\left(\frac{\alpha}{\alpha_0} \right) \tag{11}$$

in which ξ is the equivalent plastic strain, α_0 the initial value of hardening/softening and κ_c a parameter that controls the rate of hardening/softening.

In an incremental plasticity formulation the following equations hold:

$$\Delta\varepsilon_p = \Delta\lambda \frac{\partial F}{\partial \sigma}, \Delta\xi = \Delta\lambda \left\| \frac{\partial F}{\partial \sigma} \right\| \ \textit{and} \ \Delta\xi = {}^{n+1}\xi - {}^{n}\xi = \frac{1}{\kappa_c} \ln\left(\frac{{}^{n+1}\alpha}{{}^{n}\alpha} \right) \tag{12}$$

By solving Eq. 12_1 for $\Delta\lambda$, an expression for the plasticity consistency parameter as a function of the hardening/softening parameter can be derived:

$${}^{n+1}\lambda = {}^{n}\lambda + \Delta\lambda \quad \textit{with} \quad \Delta\lambda = \ln\left(\frac{{}^{n+1}\alpha}{{}^{n}\alpha} \right) \Big/ \kappa_c \left\| \frac{\partial F}{\partial \sigma} \right\| \tag{13}$$

Substituting this into Eqs (6) to (9) this system can be solved for the 4 physical parameters p, q, α and ϑ in a one or two stage fashion as mentioned earlier.

Example

To demonstrate the above described approaches, a block of material is subjected to two different loading conditions, one following the other. The first load is a prescribed linear increasing compressive strain in the vertical (Y) direction. The second is a prescribed linear increasing shear (YZ) strain. For both loads the applied strain increment is 0.001 mm/mm and the number of load steps is 50.

The material used has an E-modulus of 100 MPa and a Poisson ratio of 0.2. For the Desai yield surface the following parameters are chosen: $m = -0.5$, $n = 5.6$, $p_a = -0.1\ MPa$, $R = 0\ MPa$ and $\gamma = 0.083$. The parameter β is defined in the figures below. The hardening law as described in Eq. (11) has $\alpha_0 = 1.2 * 10^{-5}$ and $\kappa_c = 20$.

In Figs. 1 and 2 the results of the two level approach are given. In Fig. 1 the stresses are shown due to the two consecutive loading conditions for two different values of the Desai parameter β.

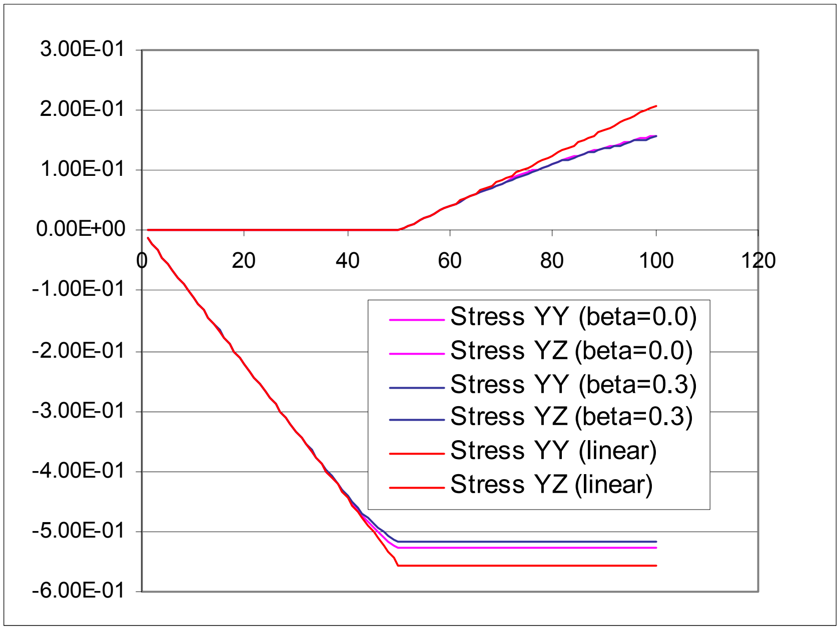

Fig. 1 Variation of stresses for the two level approach

The trajectory of the state of stress on the deviator plane for two different values of β is shown in Fig. 2.

The performance of the one-step algorithm was also evaluated by means of the same set of runs.

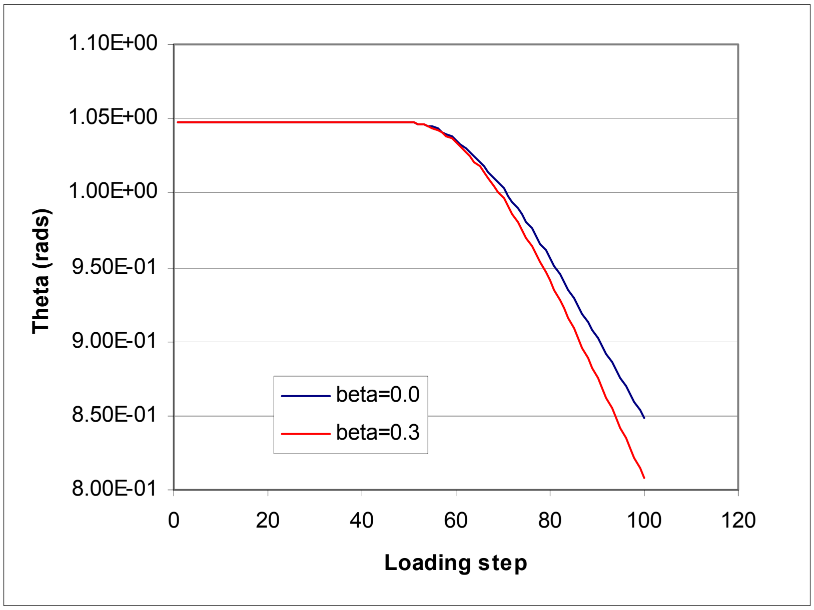

Fig. 2 Variation of Lode angle Theta for the two level approach

When the two approaches are compared, it can be concluded that, at least for the loading case described in the beginning of this section, the proposed one-step algorithm is as accurate as the two-step algorithm of Macari et al [2]. Further investigations are currently underway for several loading path conditions.

References

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