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BEM solution of 3-D dipolar gradient elastic problems in frequency domain

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Summary

A three-dimensional boundary element method (BEM) for treating time harmonic problems in linear elastic materials exhibiting microstructure effects is presented. These microstructural effects are taken into account with the aid of the dipolar gradient elastic theory of Mindlin. The constitutive equations, the boundary conditions and the integral representation of the problem are explicitly provided. Surface quadratic quadrilateral boundary elements are employed and the discretization is restricted only to the boundary. A numerical example serves to illustrate the method and demonstrate its accuracy.

Introduction

Recent experimental observations have shown that some materials are significantly affected by their microstructure and exhibit a mechanical behavior which is different than that expected classically. These microstructural effects become more pronounced especially when the size of the tested specimens becomes small as well as in cases where generated wavelengths have the same order of magnitude with the microstructure of the considered materials. Due to the lack of an internal length scale parameter the classical theory of linear elasticity fails to describe such a behavior. There are, however, other generalized continuum theories where microstructural effects are taken into account and thus materials with microstructure can be successfully modeled in a macroscopic manner. Among these theories, the most general and comprehensive theory is the one due to Mindlin [1,2] involving 16 elastic constants while a very simple dynamic version of his theory is that of dipolar gradient elasticity [3]. It is called dipolar since, besides the classical Lame' constants, only two new material constants are needed to describe the microstructural effects in the considered medium. Although, many analytical solutions of gradient elastic problems have been appeared to date in the literature, the solution of gradient elastic problems with complicated geometry and boundary conditions requires the use of numerical methods such as the finite element method (FEM) and the boundary element method (BEM). Among the efforts made for the numerical treatment of strain-gradient elastostatic boundary value problems one can mention the two-dimensional FEM solutions of Shu et al. [4], Amanatidou and Aravas [5], Teneketzis Tenek and Aifantis [6], Chen and Wang [7] and Engel et al. [8] and the two-dimensional meshless local Petrov-Galerkin (MLPG) method solution of Tang et al. [9]. In the framework of the simple gradient elastic theories with or without surface energy, one can mention the two and three-dimensional BEM solutions of Tsepoura et al. [10], Polyzos et al. [11], and

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Tsepoura et al. [12], for elastostatic problems and Tsepoura and Polyzos [13] and Polyzos et al. [14] for frequency domain elastodynamic problems.

In this work the BEM, in its direct form [15], is employed for the solution of three-dimensional elastodynamic problems in the framework of the frequency domain dipolar gradient theory. The present version of the implementation of the method is restricted to smooth boundaries and computation of boundary displacements and tractions. The constitutive equations, the equation of motion and the classical as well as the non-classical boundary conditions of a dipolar gradient elastic problem are presented. The boundary integral representation, the numerical implementation and the solution procedure of the problem are described in brief. Finally, a numerical example that illustrates the method and demonstrate its accuracy is provided.

Theoretical background

Taking into account the non-local nature of microstructural effects, Mindlin [1,2] considered that the density of strain density is not only a function of strains, as in the classical case, but also a function of the gradients of the strains. In the dipolar version of his theory, this is expressed as

$$W = \frac{1}{2}\lambda(tr\tilde{\mathbf{e}})^2 + \mu\tilde{\mathbf{e}}:\tilde{\mathbf{e}} + \frac{1}{2}\lambda g^2\nabla(tr\tilde{\mathbf{e}})\cdot\nabla(tr\tilde{\mathbf{e}}) + \mu g^2\nabla\tilde{\mathbf{e}} \,\dot{:}\, \nabla\tilde{\mathbf{e}} \tag{1}$$

where $\tilde{\mathbf{e}}$ and $tr\tilde{\mathbf{e}}$ are the classical strain tensor and its trace, respectively, ∇ represents the gradient operator, the dot, the double dots and the column of three dots indicate inner product between vectors and tensors of second and third order, respectively, (λ,μ) are the classical Lame constants and g^2 is a new material constant (units of m^2) called volumetric strain gradient energy coefficient, which correlates the microstructure with macrostructure.

Extending the idea of non-locality to the inertia of the continuum with microstructure, Mindlin proposed a new expression for the kinetic energy density function where the gradients of the velocities are taken into account, i.e.

$$T = \frac{1}{2}\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}} + \frac{1}{2}\rho\frac{h^2}{3}\nabla\dot{\mathbf{u}}:\nabla\dot{\mathbf{u}} \tag{2}$$

where ρ is the mass density, u is the displacement vector, $\dot{\mathbf{u}} = d\mathbf{u}/dt$ and h^2 is the second new material constant (units of m^2) called velocity gradient coefficient, which is always smaller than the volumetric strain gradient energy coefficient g^2. Taking the variation of strain and kinetic energy, according to the Hamilton's principle, and considering harmonic dependence on time one concludes to the equation of motion of a continuum with microstructure, which in terms of displacements is written as follows:

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} + g^2\nabla^2\left(\mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u}\right) = -\rho\omega^2\mathbf{u} + \rho\omega^2\frac{h^2}{3}\nabla^2\mathbf{u} \tag{3}$$

accompanied by the classical boundary conditions

$$\mathbf{p}(\mathbf{x}) = \hat{\mathbf{n}} \cdot \tilde{\boldsymbol{\tau}} - (\hat{\mathbf{n}} \otimes \hat{\mathbf{n}}) : \frac{\partial \tilde{\boldsymbol{\mu}}}{\partial n} - \hat{\mathbf{n}} \cdot (\nabla_S \cdot \tilde{\boldsymbol{\mu}}) - \hat{\mathbf{n}} \cdot [\nabla_S \cdot (\tilde{\boldsymbol{\mu}})^{213}] + (\nabla_S \cdot \hat{\mathbf{n}})(\hat{\mathbf{n}} \otimes \hat{\mathbf{n}}) : \tilde{\boldsymbol{\mu}}$$
$$- (\nabla_S \hat{\mathbf{n}}) : \tilde{\boldsymbol{\mu}} - \frac{\rho h^2 \omega^2}{3} \frac{\partial \mathbf{u}}{\partial n} = \mathbf{p}_0 \tag{4}$$

and/or

$$\mathbf{u}(\mathbf{x}) = \mathbf{u_0}$$

and the non-classical ones

$$\mathbf{R}(\mathbf{x}) = \hat{\mathbf{n}} \cdot \tilde{\boldsymbol{\mu}} \cdot \hat{\mathbf{n}} = \mathbf{R}_0 \quad \text{and/or} \quad \frac{\partial \mathbf{u}(\mathbf{x})}{\partial n} = \mathbf{q}_0 \tag{5}$$

where $\mathbf{p}_0, \mathbf{u_0}, \mathbf{R}_0, \mathbf{q}_0$ denote prescribed values, the symbols $\otimes$ and ∇_S indicate dyadic product and surface gradient, respectively, $\tilde{\boldsymbol{\tau}}$ is the Cauchy stress tensor and $\tilde{\boldsymbol{\mu}}$ is a third order tensor, called by Mindlin double stress tensor, related to $\tilde{\boldsymbol{\tau}}$ through the constitutive relations:

$$\tilde{\boldsymbol{\mu}} = g^2 \nabla \tilde{\boldsymbol{\tau}} \tag{6}$$

$$\tilde{\boldsymbol{\tau}} = \mu \left[\nabla \mathbf{u} + \mathbf{u} \nabla \right] + \lambda \left(\nabla \cdot \mathbf{u} \right) \tilde{\mathbf{I}} \tag{7}$$

Boundary integral representation of a 3-D gradient elastic problem

As it is proved in [16], the integral representation of the problem described in the previous section is

$$\tilde{\mathbf{c}}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) + \int_S \left\{ \tilde{\mathbf{p}}^*(\mathbf{x},\mathbf{y}) \cdot \mathbf{u}(\mathbf{y}) - \tilde{\mathbf{u}}^*(\mathbf{x},\mathbf{y}) \cdot \mathbf{p}(\mathbf{y}) \right\} dS_{\mathrm{y}} =$$
$$\int_S \left\{ \frac{\partial \mathbf{u}^*(\mathbf{x},\mathbf{y})}{\partial n_{\mathrm{y}}} \cdot \mathbf{R}(\mathbf{y}) - \tilde{\mathbf{R}}^*(\mathbf{x},\mathbf{y}) \cdot \frac{\partial \mathbf{u}(\mathbf{y})}{\partial n_{\mathrm{y}}} \right\} dS_{\mathrm{y}} \tag{8}$$

where the vector $\mathbf{p}$ represents the surface traction vector given by Eq. (4), $\mathbf{R}$ is the double traction vector given in Eq. (5) and $\tilde{\mathbf{u}}^*(\mathbf{x},\mathbf{y})$ is the fundamental solution of Eq. (3) which is given in [16] and $\tilde{\mathbf{c}}(\mathbf{x})$ is the well-known jump tensor. All the kernels appearing in the integral Eq. (8) are given explicity in [16]. Observing Eq. (8), one realizes that this equation contains three unknown vector fields, $\mathbf{u}(\mathbf{x})$ $\mathbf{p}(\mathbf{x})$ and $\partial\mathbf{u}(\mathbf{x})/\partial n$. Thus, the evaluation of the unknown fields $\mathbf{u}(\mathbf{x})$, $\mathbf{p}(\mathbf{x})$ and $\partial\mathbf{u}(\mathbf{x})/\partial n$ requires the existence of one more integral equation. This integral equation is obtained by applying the operator $\partial/\partial n_{\mathbf{x}}$ on Eq. (8) and has the form

$$\begin{aligned}\tilde{\mathbf{c}}(\mathbf{x})\cdot\frac{\partial\mathbf{u}(\mathbf{x})}{\partial n_{\mathbf{x}}}+\int_S\left\{\frac{\partial\tilde{\mathbf{p}}^*(\mathbf{x},\mathbf{y})}{\partial n_{\mathbf{x}}}\cdot\mathbf{u}(\mathbf{y})-\frac{\partial\tilde{\mathbf{u}}^*(\mathbf{x},\mathbf{y})}{\partial n_{\mathbf{x}}}\cdot\mathbf{p}(\mathbf{y})\right\}dS_{\mathbf{y}}=&\\ \int_S\left\{\frac{\partial^2\tilde{\mathbf{u}}^*(\mathbf{x},\mathbf{y})}{\partial n_{\mathbf{x}}\partial n_{\mathbf{y}}}\cdot\mathbf{R}(\mathbf{y})-\frac{\partial\tilde{\tilde{\mathbf{R}}}^*(\mathbf{x},\mathbf{y})}{\partial n_{\mathbf{x}}}\cdot\frac{\partial\mathbf{u}(\mathbf{y})}{\partial n_{\mathbf{y}}}\right\}dS_{\mathbf{y}}&\end{aligned}\tag{9}$$

The kernels appearing in Eq. (9) are given explicitly in [16]. The integral Eqs (8) and (9) accompanied by the classical and non-classical boundary conditions form the integral representation of any gradient elastic boundary value problem.

BEM solution procedure

The goal of the Boundary Element methodology is to solve numerically the well-posed boundary value problem constituted by the system of two integral equations (8) and (9) and the boundary conditions (Eqs (4) and (5)). To this end the surface S is discretized into E eight-noded quadrilateral and/or six-noded triangular quadratic continuous and discontinuous isoparametric boundary elements. Then, all the nodal fields of the corresponding fields are expressed to the local co-ordinate system ξ1, ξ2 with the aid of the shape functions Nα (α = 1, 2,…A(e) and A(e)=8 or 6 for quadrilateral or triangular elements, respectively). Adopting a global numbering for the nodes, each pair (e, α) is associated to a number β and the integral eqs (8) and (9) are written as

$$\begin{aligned}\frac{1}{2}\mathbf{u}^k+\sum_{\beta=1}^{L}\tilde{\mathbf{H}}^k_\beta\cdot\mathbf{u}^\beta+\sum_{\beta=1}^{L}\tilde{\mathbf{K}}^k_\beta\cdot\mathbf{q}^\beta&=\sum_{\beta=1}^{L}\tilde{\mathbf{G}}^k_\beta\cdot\mathbf{p}^\beta+\sum_{\beta=1}^{L}\tilde{\mathbf{L}}^k_\beta\cdot\mathbf{R}^\beta\\ \frac{1}{2}\mathbf{q}^k+\sum_{\beta=1}^{L}\tilde{\mathbf{S}}^k_\beta\cdot\mathbf{u}^\beta+\sum_{\beta=1}^{L}\tilde{\mathbf{T}}^k_\beta\cdot\mathbf{q}^\beta&=\sum_{\beta=1}^{L}\tilde{\mathbf{V}}^k_\beta\cdot\mathbf{p}^\beta+\sum_{\beta=1}^{L}\tilde{\mathbf{W}}^k_\beta\cdot\mathbf{R}^\beta\end{aligned}\tag{10}$$

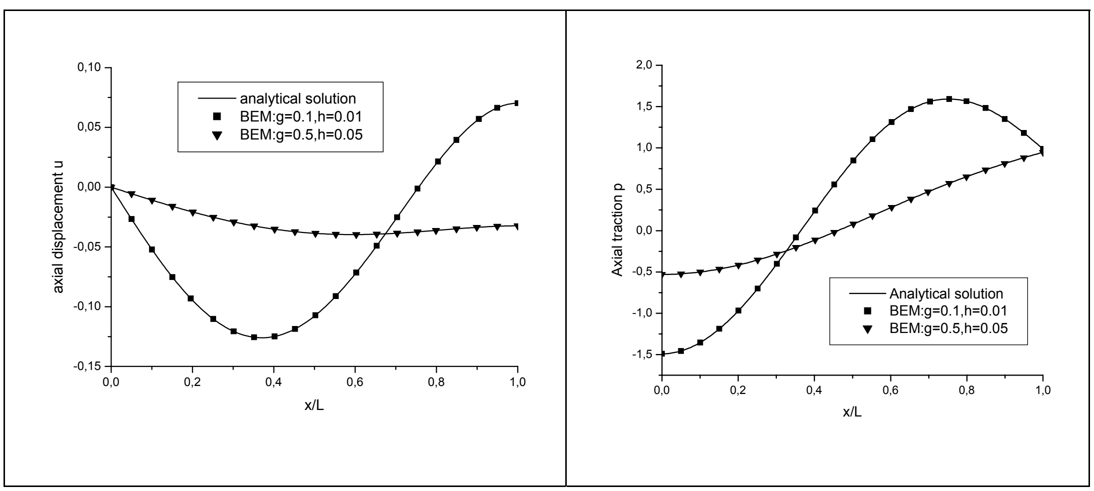
where L is the total number of nodes. Collocating Eqs (10) at all nodal points L and applying the boundary conditions (Eqs (4) and (5)) one produces the final linear system of algebraic equations of the form

$$\widetilde{\mathbf{A}} \cdot \mathbf{X} = \mathbf{B} \tag{11}$$

where the vectors **X** and **B** contain all the unknown and known nodal components of the boundary fields.

A numerical example

In order to demonstrate the accuracy of the proposed here 3-D dipolar gradient elastic boundary element methodology, a simple example dealing with the harmonic excitation of a dipolar gradient elastic cylindrical bar of length *L*, fixed at $x = 0$ and subjected to a constant axial tensile stress P_0 acting at the end $x = L$. This problem has been solved numerically utilizing a 3-D model. According to this model, the axial bar in tension is modeled by a thick solid cylinder of height *L* =*D*/4, with *D* being the diameter of the cylinder. The discretization consists of 268 quadratic quadrilateral boundary elements was restricted to one quarter of the cylinder because of symmetry. The problem has been solved for $P_0 /(\lambda + 2\mu) = 1$, and for the pairs $(g = 0.1, h = 0.01)$ and $(g = 0.5, h = 0.05)$. The dimensionless axial displacement *u*, strain and traction p have been evaluated and displayed in Fig. 1 as functions of the distance $\xi = x / L$. As it is evident, the obtained numerical results are in an excellent agreement with the analytical ones provided in [16].



References

1 Mindlin RD (1964): "Microstructure in linear elasticity" *Arch. Rat. Mech. Anal.* 10: 51-78

2 Mindlin RD (1965): "Second gradient of strain and surface tension in linear elasticity" *Int. J. Solids Struct.* 1: 417-438

3 Georgiadis HG (2003): "The mode III crack problem in microstructured solids governed by dipolar gradient elasticity: Static and dynamic analysis" *Journal of Applied Mechanics-Transactions of the ASME* 70 (4): 517-530.

4 Shu JY, King WE, Fleck NA (1999): "Finite elements for materials with strain gradient effects" *Int. J. Num. Meth. Engng* 44: 373-391.

5 Amanatidou E, Aravas N (2002): "Mixed finite element formulations of strain-gradient elasticity problems" *Comput. Methods Appl. Mech. Engng* 191: 1723-1751.

6 Teneketzis Tenek L, Aifantis EC (2002): "A two-dimensional finite element implementation of a special form of gradient elasticity" *Comp. Model. Engng. Sci*. 3: 731-741.

7 Chen SH, Wang TC (2002): "Finite element solutions for plane strain mode I crack with strain gradient effects" *Int. J. Solids Struct.* 39: 1241-1257.

8 Engel G, Garikipati K, Hughes TJR, Larson MG, Mazzei L, Taylor RL (2002): "Continuous/discontinuous finite element approximations of fourth-order elliptic problems in structural and continuous mechanics with application to thin beams and plates, and strain gradient elasticity" *Comput. Methods Appl. Mech. Engrg.* 191: 3669-3750.

9 Tang Z, Shen S, Atluri SN (2003): "Analysis of materials with strain-gradient effects: A Meshless Local Petrov-Galerkin (MLPG) approach with nodal displacements only" *CMES: Computer Modeling in Engineering & Sciences* 4: 177-196.

10 Tsepoura KG, Papargyri-Beskou S, Polyzos D (2002): "A boundary element method for solving 3D static gradient elastic problems with surface energy" *Comput. Mech.* 29: 361-381.

11 Polyzos D, Tsepoura KG, Tsinopoulos SV, Beskos DE (2003): "A boundary element method for solving 2-D and 3-D static gradient elastic problems PartI: Integral formulation" *Comput. Methods Appl. Mech. Engrg* 192: 2845-2873.

12 Tsepoura KG, Polyzos D, Tsinopoulos SV, Beskos DE (2003): "A boundary element method for solving 2-D and 3-D static gradient elastic problems PartII: Numerical implementation" *Comput. Methods Appl. Mech. Engrg* 192: 2875-2907

13 Tsepoura KG, Polyzos D (2003): "Static and harmonic BEM solutions of gradient elasticity problems with axisymmetry" *Comput. Mech.* 32: 89-103.

14 Polyzos D, Tsepoura KG, Beskos DE (2003): "BEM solutions of frequency domain gradient elastodynamic 3-D problems" *Electronic Journal of Boundary Elements*.

15 Beskos DE (1997): “Boundary element methods in dynamic analysis, Part II (1986-1996)” *Appl. Mech. Rev ASME* 50: 149-197.

16 Polyzos D (2004): “Frequency domain BEM for solving dipolar gradient elastic problems” *Comput. Mech.* To appear.