Transient behavior of scattered and reflected wave fields in anisotropic solids

S. Hirose[1], A.Tan[2], Ch. Zhang[3]

Summary

Elastodynamic wave fields for anisotropic solids are presented in this paper. The wave fields due to a cavity or a crack in unbounded anisotropic media are generated. To reconstruct the scattered waved fields, two dimensional time domain boundary element methods are developed. A displacement boundary integral formulation is used to solve the displacements at points along the cavity while a traction boundary integral equation is formulated for a crack to obtain the unknowns crack opening displacements (CODs). Deformation wave fields are then rendered using displacement BIE's. The BIE formulations use the fundamental solutions derived by Wang and Achenbach. Both BIEs require higher order fundamental solutions. Numerical results for the scattering of elastic waves for a crack and a cavity are shown.

Introduction

The boundary element method is used as a computational tool to predict the effects of the waves on materials. The study of elastic wave scattering usually finds applications in ultrasonic testing and evaluation of materials. The prediction of the reflected and scattered waves provides essential features for defect characterization.

In this paper we aim to present elastodynamic wave propagation in anisotropic media with defect. The defects are characterized by either a crack or a cavity. Two boundary element methods are used. First, a displacement BIE in two dimensional elastic and anisotropic region is formulated. Second, the scattering of the elastic wave by a traction free crack in a two dimensional elastic and anisotropic region is considered. The crack opening displacements (COD's) are obtained as the solution of the traction boundary integral method. Fundamental solutions derived by Wang and Achenbach are used. The static fundamental parts are composed of relatively simple explicit expressions in closed form[1]. The dynamic parts are in terms of line integrals over a unit circle

[1] Dept. Mech. & Environ. Informatics, Tokyo Institute of Technology, 2-12-1 W8-22, O-okayama, Meguro-ku, Tokyo, 152-8552, Japan

[2] Dept. Civil Eng'g, Tokyo Institute of Technology, 2-12-1 O-okayama, Meguro-ku, Tokyo, 152-8552, Japan

[3] Dept. Civil Eng'g, University of Applied Sciences Zittau/Görlitz, Theodor-Körner-Allee 16, D-02763, Zittau

which is due to the inverse Radon transform used to derive the fundamental solutions[1].

Fundamental Solutions

In the absence of body forces, the solid satisfies the equations of motion and the Hooke's law where we assume two dimensional displacement and stress fields in plane strain state independent of the x_3-coordinate.

$$\sigma_{i\beta,\beta} = \rho\ddot{u}_i \tag{1}$$

$$\sigma_{i\beta} = C_{i\beta k\lambda}u_{k,\lambda} \tag{2}$$

The Green's function is defined as the solution of the following equations:

$$\left\{\Gamma_{pi}(\partial_1,\partial_2) - \rho\delta_{pi}\partial_t^2\right\} g_{ij}(\boldsymbol{x};t) = -\delta_{pj}\delta(\boldsymbol{x})\delta(t) \tag{3}$$

$$g_{ij}(\boldsymbol{x};t) = 0, \qquad \text{for } t < 0$$

where $\Gamma_{pi}(\partial_1,\partial_2) = C_{i\beta k\lambda}\partial_\beta\partial_\lambda$. Wang and Achenbach [1] using Radon transform derived explicit expressions of the Green's functions. Due to inverse Radon transform the Green's function g_{pk} includes an integrand over a unit circle.

$$g_{pk}(\boldsymbol{x},t) = \frac{H(t)}{4\pi^2}\int_{|n|=1}\sum_{m=1}^{M}\frac{P_{pk}^m}{\rho c_m}\left(\frac{d\boldsymbol{n}}{c_m t + \boldsymbol{n}\cdot\boldsymbol{x}}\right) \tag{4}$$

where $P_{pk}^m = E_{pk}^m / E_{qq}^m$, and E_{pk}^m and ρc_m^2 are the eigenvectors and eigenvalues for the matrix $\Gamma_{pi}(n_1,n_2)$.

The Green's function can be reformulated into two parts, namely static and regular part by applying integration by parts. The singular part corresponds to the elastostatic Green's function which has a closed form solution. Higher order Green's functions are needed for the BIEs namely:

$$h_{ik}\left[\boldsymbol{x}-\boldsymbol{y},\boldsymbol{e}(\boldsymbol{x});t\right] = C_{i\gamma p\delta}e_\gamma(\boldsymbol{x})\frac{\partial}{\partial x_\delta}g_{pk}(\boldsymbol{x}-\boldsymbol{y};t) \tag{5}$$

$$w_{ij}\left[\boldsymbol{x}-\boldsymbol{y},\boldsymbol{e}(\boldsymbol{x}),\boldsymbol{e}(\boldsymbol{y});t\right] = C_{i\gamma p\delta}e_\gamma(\boldsymbol{x})\frac{\partial}{\partial x_\delta}C_{j\alpha k\beta}e_\alpha(\boldsymbol{y})\frac{\partial}{\partial y_\beta}g_{pk}(\boldsymbol{x}-\boldsymbol{y};t) \tag{6}$$

Since the closed form expression of the elastostatic Green's function is available, the higher order static parts can also be reduced to closed form expressions by application of Stroh's formalism [2].

Cavity in anisotropic solid

In scattering problems the total wave field is written as the superposition of the scattered field and the incident field.

$$u_i = u_i^{in} + u_i^{sc}, \qquad t_i = t_i^{in} + t_i^{sc} \tag{7}$$

Since no scattering exists before the incident wave hits a scatterer, initial scattered waves are zero $u_i^{sc}(\boldsymbol{x},t)=0, \quad \text{for } t \le 0$.

Consider a cavity with a boundary S in a homogeneous, linearly elastic anisotropic solid D. Since the cavity has zero traction, we have

$$u_k^{in}(\boldsymbol{y},t) - \int_S h_{ik}\left[(\boldsymbol{x}-\boldsymbol{y}), e(\boldsymbol{x}); t\right] * u_i(\boldsymbol{x},t)d\boldsymbol{x} = \begin{cases} u_k(\boldsymbol{y},t) & \boldsymbol{y} \in \boldsymbol{D} \\ u_k(\boldsymbol{y},t)/2 & \boldsymbol{y} \in \boldsymbol{S} \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

The scattered wavefield is derived from eq (8) and applying eq (7) with the assumption of zero traction in the cavity.

$$u_j^{sc}(\boldsymbol{y},t) = -\int_S h_{ij}\left[(\boldsymbol{x}-\boldsymbol{y}), e(\boldsymbol{x}); t\right] * u_i(\boldsymbol{x},t)d\boldsymbol{x} \qquad \boldsymbol{y} \in \boldsymbol{D} \tag{9}$$

Finite crack in anisotropic solid

Consider a homogeneous, linearly elastic anisotropic solid containing a crack of finite shape with zero traction. The boundary integral equation for a crack problem can be formulated from elastodynamic reciprocal theorem of Betti-Rayleigh [3]. The boundary integral equation is written as follows:

$$u_k(\boldsymbol{y},t) = u_k^{in}(\boldsymbol{y},t) - \int_{\partial D} h_{ik}\left[(\boldsymbol{x}-\boldsymbol{y}), e(\boldsymbol{x}); t\right] * \Delta u_i(\boldsymbol{x},t)d\boldsymbol{x} \tag{10}$$

where Δu_i is the crack opening displacement. Eq (10) is then substituted to eq (2) obtaining the traction boundary integral equation for elastic wave propagation.

$$t_j^{in}(\boldsymbol{y},t) - \int_{\partial D} w_{ij}\left[(\boldsymbol{x}-\boldsymbol{y}), e(\boldsymbol{x}), e(\boldsymbol{y}); t\right] * \Delta u_i(\boldsymbol{x},t)d\boldsymbol{x}$$
$$= \begin{cases} t_j(\boldsymbol{y},t) & \boldsymbol{y} \in \boldsymbol{D} \\ 1/2\left[t_j(\boldsymbol{y}^+,t) + t_j(\boldsymbol{y}^-,t)\right] & \boldsymbol{y} \in \partial\boldsymbol{D} \\ \boldsymbol{0} & \text{otherwise} \end{cases} \tag{11}$$

where $\boldsymbol{y}^{+}$ and $\boldsymbol{y}^{-}$ indicate a pair of points on the crack surface with one on each side. Since the crack has no traction, i.e., $1/2\left[t_j(\boldsymbol{y}^{+},t)+t_j(\boldsymbol{y}^{-},t)\right]=0$, eq (11) is the boundary integral equation with the unknown crack opening displacement.

The scattered deformation fields around the crack can be calculated using the following equation.

$$u_j^{sc}(\boldsymbol{y},t)=-\int_{\partial D} h_{ij}\left[(\boldsymbol{x}-\boldsymbol{y}),e(\boldsymbol{x});t\right]*\Delta u_i(\boldsymbol{x},t)d\boldsymbol{x} \tag{12}$$

Numerical Examples

Numerical computations have been carried out for a plane transient wave in an infinite transversely isotropic solid. Comparison between the crack and cavity scattering are shown. To closely monitor the results a flat cavity with an elliptical shape of length 2a and aspect ratio of $b/a=0.01$. For the cavity, the boundary is divided into 80 elements with its origin at the center point. The crack is divided into 41 elements with length of 2a. Non-zero elastic constants for the material are given as follows: $c_{11}/c_{66}=c_{33}/c_{66}=45.91$, $c_{12}/c_{66}=c_{23}/c_{66}=1.84$, $c_{22}/c_{66}=3.98$, $c_{44}/c_{66}=1$, $c_{55}=(c_{33}-c_{13})/2$. The x_2-axis is taken as the axis of symmetry for the transversely isotropic solid [4]. A vertical incidence of the longitudinal wave is considered and with waveform given by a triangular pulse as

$$u_2^{in}(x,t)=(u_0/\delta)\begin{bmatrix} p_1H(p_1)-2(p_1-\delta)H(p_1-\delta) \\ -(p_1-2\delta)H(p_1-2\delta) \end{bmatrix} \tag{13}$$

here $p_1=t-c_1^{-1}(x_2+a)$ and $\delta=0.25a/c_1$. The wave velocities are computed as $c_2=\sqrt{c_{66}/\rho}$ and $c_1/c_2=1.994$. u_0 is the amplitude of the triangular pulse and 2δ the base of the pulse.

Figure 1 shows images of scattered field due to the cavity. Figure 2 shows the effect due to the crack. The figures show similar results. After the incident wave hits the defect, reflected waves by the crack face propagate in both forward and backward directions. Sideways scattering by the crack tip are seen because of higher wave speed in the x_1-direction. Sideways scattered waves are seen moving away from the tips and the reflected fields moves away from the defect. The quasi-transverse wave diffracted by the crack tip can be seen developing at time 0.25. These pseudo transverse waves produce an expanding wave front moving away from the crack tip. Due to the presence of the crack larger deformation of these waves can be seen in the wave front moving towards the center of the crack.

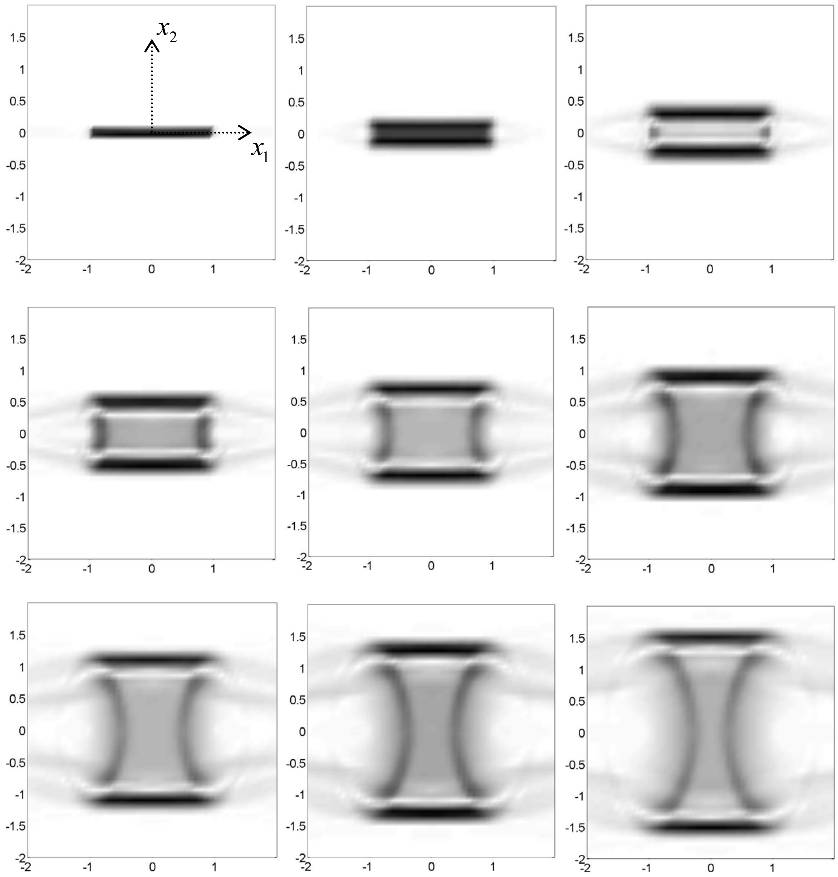

Fig.1. Scattered wave field due to an elliptical cavity for plane longitudinal wave at times: $\tau = \sqrt{c_{66}/\rho}\, t/a =$ 0.05, 0.15, 0.25, 0.35, 0.45, 0.55, 0.65, 0.75, and 0.85.

Conclusions

Two dimensional time-domain boundary element method has been developed to generate the scattered wave fields in elastodynamic anisotropic solids with defect. Elastic scattering of waves by a crack and by a cavity have been numerically calculated. Wave fields are rendered for a transversely isotropic solid. Numerical examples demonstrated the interesting features of scattering and reflection of waves in an anisotropic solid.

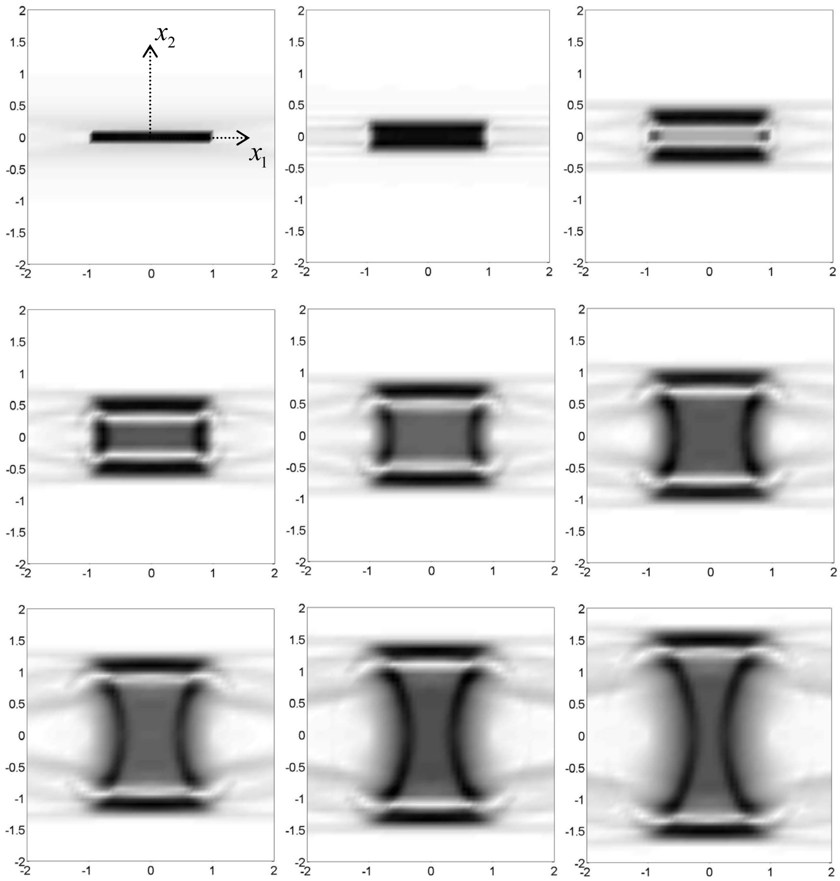

Fig.2. Scattered wave field due to a crack for plane longitudinal wave at times: $\tau = \sqrt{c_{66}/\rho}\, t/a =$ 0.05, 0.15, 0.25, 0.35, 0.45, 0.55, 0.65, 0.75, and 0.85.

Reference

1 Wang C.Y. and Achenbach, J.D. (1994): `` Elastodynamic fundamental solutions for anisotropic solids '', *Geophys.J. Int.*, Vol. 118, pp. 384-392.

2 Hirose, S., (1999): `` Dynamic Analysis for a crack in anisotropic elastic solid'',*JASCOME*, Vol. 16, pp. 13-17. (in Japanese)

3 Zhang Ch. and Gross D., (1998): *On Wave Propagation in Elastic Solids with Cracks*, Computational Mechanics Publications, Southampton UK.

4 Wang C.Y. , Achenbach, J.D. and Hirose S., (1996): `` Two dimensional time domain BEM for scattering of elastic waves in solids of general anisotropy '', *International Journal of Solids and Structures*, Vol. 33, no.26, pp. 3843-3864.