Analysis of Elliptical Waveguides by Method of Fundamental Solutions

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Summery

The present work describes the application of the method of fundamental solutions (MFS) for the solution of cutoff wavelengths of elliptical waveguides. Since the MFS employs a formulation using the boundary values only, the cutoff wavelengths are determined by applying the singular value decomposition (SVD) technique. The use of the MFS to solve the governing equation（Helmholtz equation）guarantees solution without singularities since it does not use discretized points to determine the solution at the interior of the computational domain. The combination of MFS and SVD technique has resulted in a simpler and efficient numerical solution procedure as compared to the other numerical schemes.

Introduction

Elliptical waveguides have been widely applied in many engineering fields. Chu [1] presented the theory of the transmission of electromagnetic waves in hollow conducting pipes of elliptic cross section and reported numerical results for the cutoff frequency and the attenuation for six waves. It is very crucial to determine the cutoff wavelengths of an elliptical waveguide for designing the waveguides and hence this field has attracted many researchers in the recent past. Zhang and Shen presented analytical solutions of the elliptical waveguides [2], and most of the attention was paid to computing the zeros of the modified Mathieu functions of the first kind. Lately, a meshless collocation method with the Wendland radial basis functions was conducted by Jiang et al. [3].

Meshless methods are becoming popular in recent decade and many efforts have been devoted to the development of meshless methods by many researchers [4 & 5]. In the present work a new numerical method based on the method of fundamental solutions (MFS) is devised to solve the cutoff wavelengths of elliptical waveguides.

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Unlike the FDM or FEM the present method does not use any domain discretization scheme to obtain the grid points inside the computational domain and hence MFS is normally considered as a meshless numerical method. In the MFS, only the equations governing the boundary node points are solved, thus saving a lot of computational time and effort as compared to other numerical schemes. The solutions at the interior domain are obtained just by means of simple interpolation based on the MFS. In the present work, the MFS is applied to solve an elliptical waveguide problem with an aim to determine the cutoff wavelengths for transverse electric (TE) and transverse magnetic (TM) modes. The details about the governing equations and numerical methods used are discussed in section II and III. A discussion on the predicted results is provided in section IV. The conclusions arrived based on the present work are discussed in section V.

Governing Equations and Boundary Conditions

The transverse electric (TE) and transverse magnetic (TM) waves are defined using the eigenfunctions of the Helmholtz equation with both Dirichlet and Neumann boundary conditions as follows:

$$\nabla^2\phi + k_c^2\phi = 0 \qquad (1)$$

where k_c is the cutoff wavenumber. For the case of TM waves, $H_z = 0$, $\varphi = E_z$, while for the TE waves, $E_z = 0$, $\varphi = H_z$. The TM waves should satisfy the following Dirichlet boundary condition:

$$\phi\big|_\Gamma = E_z\big|_\Gamma = 0 \qquad (2)$$

The TE waves satisfy the Neumann boundary condition given as follows:

$$\left.\frac{\partial\phi}{\partial \mathrm{n}}\right|_\Gamma = \left.\frac{\partial H_z}{\partial \mathrm{n}}\right|_\Gamma = 0 \qquad (3)$$

where Γ refers the boundary of the waveguide and is written as the following equation for an ellipse:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \qquad (4)$$

Here, the parameters a and b are semi major axis and semi minor axis respectively.

The purpose of this study is to solve the eigenvalue k_c of the resulting matrix obtained from the Helmholtz equation. Once the eigenvalues are known then the cutoff wavelengths of the waveguides can be computed. The application of the MFS to solve the Helmholtz equation for the TM and TE modes are discussed in the following section.

Applications of MFS

The MFS makes use of the field variables known at the boundary to obtain the solution for the interior of the domain. From the principle of the MFS, for a given governing equation, the free space Green's function has to be satisfied. In the present cases, the free space Green's function is written for the Helmholtz equation as follows:

$$\nabla^2 G(\vec{x}) + k_c^2 G(\vec{x}) = -\delta(\vec{x} - \vec{\xi}) \tag{5}$$

where $G(r_i) = \frac{-i}{4} H_0^{(2)}(k_c r_i) = \frac{-i}{4}\left[J_0(k_c r_i) - iY_0(k_c r_i)\right]$ is the fundamental solution, $\delta(\vec{x} - \vec{\xi})$ is the Dirac delta function, $\vec{x}$ is the position of the field point, and $\vec{\xi}$ is the position of the source point and H_0, J_0,and Y_0 are the Hankel and Bessel functions. Using the above expression, the approximate solution for the Helmholtz equation is obtained as：

$$\phi\left(\overrightarrow{x_i}\right) = \sum_{j=1}^{N} \alpha_j^{\phi} G\left(r_{ij}\right) \tag{6}$$

where N is the number of boundary points. Finally, using method of collocation to satisfy the boundary condition and SVD technique to obtain the cutoff wave number, we can obtain the coefficients and interior field of TE and TM modes.

Numerical Results and Discussions

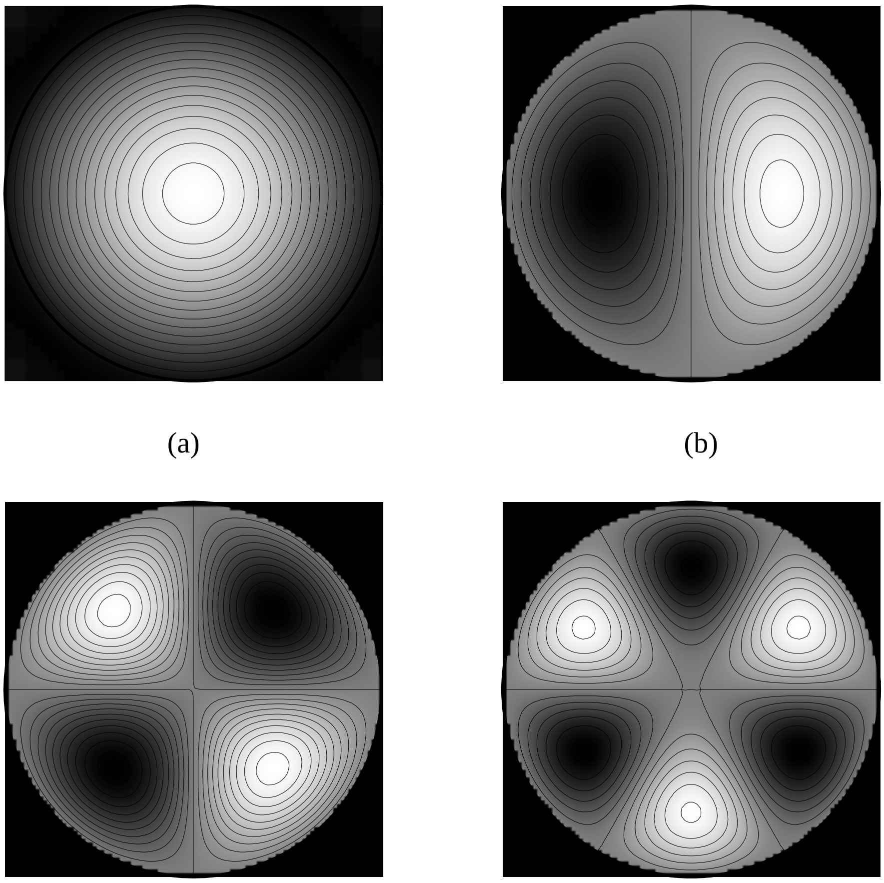
The determination of cutoff wavelength of elliptical wavegudes in TM and TE modes, as considered by Jiang et al. [3] and the analytical solutions [2] is taken as a test problem in the present work. Initially the application of the MFS to solve the Helmholtz equation results in a singular matrix as given by the equation. The eigenvalues are first obtained by solving the singular matrix using the SVD technique. The cutoff wavelengths for the TM mode with e=0.1 are solved by using the SVD technique, and these results are tabulated along with the results of analytic solutions [2] and Jiang et al. [3] as shown in Table 1. The results obtained with boundary point N=25 give almost the same results as that of boundary point N=20 but at the same time agreeing closely with the analytical results [2]. The phenomena of vibrating eigenmodes of TM and TE mode are shown in Figure 1. Good results are also obtained for the corresponding TE mode, though they are not shown here due to the look of space.

Table 1 Normalized Cutoff Wavelengths of TM Modes for e= 0.1

	Analytical solutions (Ref[2])	MFS		RBF(Ref[3])	
		N=20	N=25	N=49	N=171
1	2.6062	2.6062	2.6062	2.6081	2.6062
2	1.6377	1.6377	1.6377	1.6435	1.6378
3	1.6336	1.6336	1.6336	1.6393	1.6337
4	1.2204	1.2204	1.2204	1.2337	1.2205
5	1.2204	1.2204	1.2204	1.2334	1.2204
6	1.1353	1.1353	1.1353	1.1552	1.1353
7	0.9823	0.9823	0.9823	1.0197	0.9823
8	0.9823	0.9823	0.9823	1.0196	0.9823
9	0.8945	0.8945	0.8945	0.9490	0.8945

Conclusions

The two-dimensional homogeneous Helmholtz equation governing the behavior of the cutoff wavelengths of elliptical waveguides has been solved using the MFS along with the SVD technique. The ability of the MFS to compute the eigenvalues and eigenmodes of TM and TE modes for an elliptical waveguide has been tested successfully. The demonstrated good performance of the MFS scheme shows that it a powerful tool for the numerical solution of elliptical waveguides problem.



(a) (b)

(c) (d)

Figure.1 The phenomena of TM mode for e=0.1
（a）k_c =2.4（b）k_c =3.8（c）k_c =5.1（d）k_c =6.4

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