

A numerical scheme for post-buckling analysis of thin-walled members in the context of GBT

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Summary

In recent years, the post-buckling behavior of thin walled cold-formed steel members under compressive or bending loads has been the object of several research studies. For the most relevant situations in practice the problem is assumed to be conservative, so that the system's behavior can be described from a total potential energy (TPE) function. In order to apply the Rayleigh-Ritz method, a discretization procedure is needed and the TPE becomes a function of a load parameter P and of a set of n_C generalized coordinates a_i . The post-buckling analysis concerns the search of alternative equilibrium paths in the neighborhood of the critical states – the one associated with the lowest critical load parameter having physical interest. The paper presents a set of numerical techniques required to study the stability behavior of thin-walled members in the context of an extended formulation of GBT (Generalized Beam Theory). An application to the study of the equilibrium and buckling behavior of cold-formed members is also presented.

Introduction

The GBT (Generalized Beam Theory) was invented in the sixties by Schardt [1] and it is based on the characterization of the response of a prismatic thin-walled member – whose dimensions and displacements are show in Fig. 1 – as a linear combination of pre-established cross section deformation patterns, henceforth called modes of deformation. Within the scope of an extended GBT formulation recently developed by the authors [2], there are five types of modes of deformation, each type related to the relevant deformation pattern considered in the mode: (i) main nodes warping modes, (ii) plate bending modes, (iii) inner nodes warping modes, (iv) plate transversal extension modes, and (v) plate distortional modes. After an orthogonalization procedure [1,2], these basic modes of deformation are combined in such a way that some matrices of the linear equilibrium system become diagonal, thus giving a mechanical meaning to some modes of deformation. For example, the 1st mode becomes the axial elongation and the 2nd and the 3rd ones become the bending modes about the principal axes. The resulting modes of deformation associated with no transversal bending neither transversal extension of the plates are called rigid body modes, while the remaining ones are designated by distortional modes. The relevant displacements are computed as [1]:

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$$u(x,s) = \sum_{k=1}^{n_{MD}} {}^k u(s) \cdot {}^k V'(x) \quad (1,a-c)$$

$$f(x,s) = \sum_{k=1}^{n_{MD}} {}^k f(s) \cdot {}^k V(x)$$

$$m_s(x,s) = \sum_{k=1}^{n_{MD}} {}^k m_s(s) \cdot {}^k V(x)$$

where ${}^k u(s)$, ${}^k f(s)$ and ${}^k m_s(s)$ are the displacements and transverse moments pre-established unitary configuration for mode k and ${}^k V(x)$ is the k -mode amplitude function, which depends only on the longitudinal coordinate x and is yet unknown (note that all displacements in the cross section plane and the transversal bending moments are a function of ${}^k V$, while the longitudinal displacements depend of ${}^k V'$ [1]).

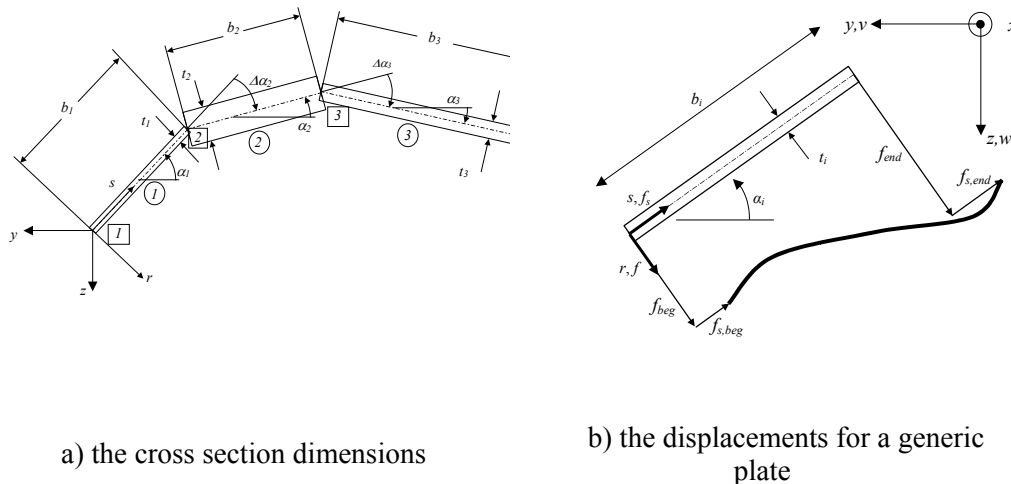


Fig. 1 – Definition of dimensions and displacements for the thin-walled cross section

In order to deal with all modal interaction phenomena between local plate behavior, distortional behavior and global response (flexural, torsional and flexural torsional), a GBT energy formulation [2,3] was derived and applied to the study of thin walled members with open or closed cross section in the post-buckling range [2,4].

It is the purpose of the present paper to present the numerical strategies used to analyze the equilibrium of a thin walled member in the context of a GBT energy formulation, mainly in the post-buckling domain. Finally, an application to the stability analysis of a compressed thin-walled RHS column is presented.

The TPE, the discretization procedure and the equilibrium system

From the classical definition of internal strain energy,

$$U_i = \int_0^L \int_A \left(\frac{1}{2} \sigma_x^M \varepsilon_x^M + \frac{1}{2} \tau_{sx}^M \gamma_{sx}^M + \frac{1}{2} \sigma_x^B \varepsilon_x^B + \frac{1}{2} \sigma_s^B \varepsilon_s^B + \frac{1}{2} \tau_{sx}^B \gamma_{sx}^B \right) dA dx, \quad (2)$$

and potential energy of the external loading at a generic section $x = \bar{x}$,

$$\Pi = \sum_{k=1}^{n_{MD}} \left(Q_{r,x=\bar{x}} \cdot {}^k f + Q_{s,x=\bar{x}} \cdot {}^k f_s \right) {}^k V|_{x=\bar{x}} + Q_{x,x=\bar{x}} \cdot {}^k u \cdot {}^k V'|_{x=\bar{x}}, \quad (3)$$

where Q_r , Q_s and Q_x denote, respectively, the loading through the Or , Os and Ox axis, one obtains the total potential energy A as a function of the load parameter P and the n_{md} amplitude modal functions ${}^k V$ [2]. In order to analyze the member's equilibrium the Rayleigh-Ritz method is applied, approximating each amplitude modal function ${}^k V$ by a linear combination of pre-established coordinate functions ${}^k g_i$:

$${}^k V = {}^k a_1 {}^k g_1 + {}^k a_2 {}^k g_2 + {}^k a_3 {}^k g_3 + K \quad (4)$$

coefficients ${}^k a_i$ becoming the unknowns of the problem and henceforth being designated by generalized coordinates of the system (a global numbering can be used and so subscript i may disappear). In the context of the present work, the functions ${}^k g_i$ are polynomials in x and are computed through a sequential procedure that derives the coordinate functions from the relevant modal boundary conditions, based on the concepts of orthogonality and normalization of functions [4]. The TPE can thus be defined as:

$$A = A\left({}^l a, K, {}^{n_c} a, P\right) \quad (5)$$

where n_c denotes the total number of generalized coordinates, and P represents the external control parameter, appearing in (5) in a linear form.

It is taken as axiomatic [5] that equilibrium states are associated with stationary values of A with respect to the generalized coordinates, and stable equilibrium states are related to complete relative minimums of the TPE with respect to the generalized coordinates. So, the equilibrium system is defined as:

$$F_i\left({}^l a, K, {}^{n_c} a, P\right) = \frac{\partial A}{\partial {}^i a} = 0, \quad i=1, \dots, n_c \quad (6)$$

where F_i are polynomials in ${}^i a$ and P , and the concepts related to the classical problem of the succession of shapes in a system can be applied [6]. Therefore, the problem consists in determining the values of ${}^i a$ and P that satisfy system (6) and, subsequently, at each point in the ${}^i a$ - P space that satisfies (6), in analyzing the stability of the corresponding equilibrium state.

From the unloaded state to the critical state

Due to the physical properties of the system, the unloaded state, defined by:

$$P = 0 \text{ and } {}^i a = 0, i=1, \dots, n_C, \quad (7)$$

is a particular and stable solution of (6), and the equilibrium path emerging from it - the fundamental path (FP) - is derived through the power series method [7] by assuming that it can be expressed as a Taylor expansion in the neighborhood of the unloaded state in the following form (for most situations in practice this assumption is not restrictive, since the FP is almost linear in the range of interest):

$${}^i a_{FP}(\lambda) = {}^i a_{(1)}P + {}^i a_{(2)}P^2 + {}^i a_{(3)}P^3 + K, \quad i=1, \dots, n_C. \quad (8)$$

where coefficients ${}^i a_{(j)}$ are yet unknown. Expression (8) is then introduced into the equilibrium system (6) and all coefficients ${}^i a_{(j)}$ are determined by rendering to zero all coefficients of the powers in P , since along an equilibrium path all polynomials F_i of (6) must be identically equal to zero. In practice, expansions until third order terms enable sufficient precision for most situations. The FP will intersect other equilibrium paths - post-buckling paths, yet unknown - at points of bifurcation, which are to be found.

Associated with FP, a coordinate transformation is introduced in the form [5,8,9]:

$${}^i a = {}^i a_{FP}(P) + {}^i q, \quad i=1, \dots, n_C, \quad (9)$$

this transformation being valid outside the regions of the FP lying in the neighborhood of limit points - for thin-walled prismatic members under compression or bending this assumption is not restrictive because, as it was said above, the FP is usually almost linear in the range of interest. It is very important to remember that the FP is given, in the ${}^i q$ - P space, as a trivial solution in terms of the sliding coordinates ${}^i q$. Afterwards, a new TPE function W is defined in terms of the sliding coordinates as:

$$W({}^i q, P) = A({}^i a_{FP}(P) + {}^i q, P), \quad (10)$$

and, in general, the linearity in P disappears in expression (10) [5]. The equilibrium and the stability conditions pass over unchanged to the W function, with respect to the generalized coordinates ${}^i q$ [8].

The fundamental path remains stable until it reaches a critical point, which is associated with the zeroing of the second variation of W with respect to the generalized coordinates. The critical points along the FP are thus found by rendering to zero the determinant of the Hessian matrix of W along the FP:

$$\det(\mathbf{H}_{FP}) = \left| \frac{\partial^2 W}{\partial {}^i q \partial {}^j q} \right|_{FP} = 0. \quad (11)$$

Due to the shape of \mathbf{H}_{FP} [3], equation (11) can be presented in the following form:

$$[\mathbf{H}_{FP}] \{q\} = 0 \Leftrightarrow [\mathbf{H}_{PF}^{(0)} + \mathbf{H}_{PF}^{(1)}P + \mathbf{H}_{PF}^{(2)}P^2 + \mathbf{H}_{PF}^{(3)}P^3 + \mathbf{K}] \{q\} = 0, \quad (12)$$

giving rise to a non-linear eigenproblem that can be solved through the UISM technique [10], thus enabling the calculation of the eigenvalues, which are the critical loads – the lowest one having the most relevant physical interest – and their corresponding eigenvectors – the critical modes. The lowest eigenvalue, P_{CR} , and its correspondent eigenvector, $\{q\}_{CR}$, define the relevant critical state around which the post-buckling equilibrium paths will be searched. Also, through expression (8), P_{CR} defines the values of all coordinates $^i a$ at the critical state, henceforth called $^i a_{CR}$, and $\{q\}_{CR}$ determines, in a unique way, the set of active and passive coordinates at the critical state: its n_A non-zero coefficients – each one, after a normalization procedure, is a measure of the participation of the corresponding coordinate in the bifurcational phenomenon – correspond to the set of active coordinates (thus defining the co-rank n_A [6]), and the remaining ones become the passive coordinates, thus defining the rank of the critical state as $n_C - n_A$.

Searching post-buckling equilibrium paths near the critical state

The classical procedures for searching post-critical equilibrium paths in perfect elements [5,8,9,11] are based on perturbation methods [12] that imply the definition of the post-buckling equilibrium paths as Taylor expansions around the critical state. This procedure becomes very cumbersome if a high number of coordinates is required and introduces a sometimes restrictive assumption to the post-buckling analysis of structural members subjected to bending and/or compression, by imposing to each coordinate along the post-buckling equilibrium paths to be a function of an adopted control parameter [13]. In the present work an alternative procedure is applied, based only in the definition of an equilibrium path – any path that respects the equilibrium system (6). Among all equilibrium paths, one searches the one that intersects the FP at the relevant critical state determined in the above paragraph. So, from the above paragraph some information about this path can be obtained: this post-buckling equilibrium path (PBP) contains the critical state defined by $^i a_{CR} - P_{CR}$, and it is already known which coordinates are active - these will bifurcate in the post-buckling domain. Let's denote $^j a$ the most participative coordinate. The task is simply to track the PBP in the $^i a - P$ space in the neighborhood of the critical state $^i a_{CR} - P_{CR}$, as it can be seen in Fig. 2-a) for a simple 3D case. Adopting the relevant numerical techniques for solving non-linear equations systems [14,15] and expressing all coordinates in their transformed shape (9), one starts from the hyper-surface normal to the hyper-plane $^j a - P$ that contain the FP, and moves this hyper-surface to both sides by imposing non-zero values to $^j q$, as it can be seen in fig. 2-b). Then, one simply finds the points near the critical state that respect equilibrium system (6) and that belong to the hyper-surfaces thus generated – see points *A*, *B* and *C* in Fig. 2-a). The unknowns of the resulting equation system are then all sliding coordinates $^i q$ (with the

exception of ${}^j q$, whose value was imposed) and the load parameter P . After having found the first non-fundamental equilibrium point, it is possible to opt between proceeding with this scheme, which forces the equilibrium points to be outside of the FP and is valid as long as the equilibrium points move away from the FP in the ${}^j a$ - P hyper-plane, or by any other path-searching method [16].

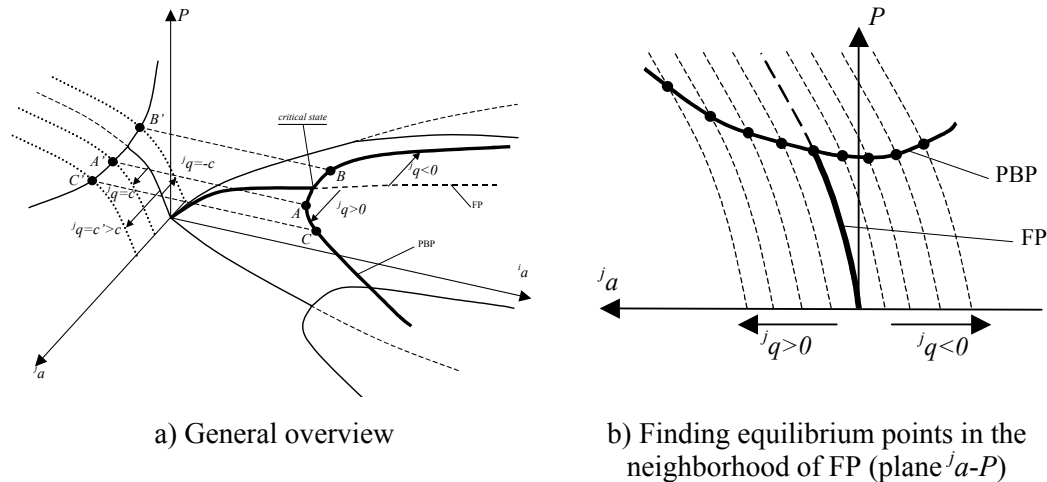


Fig. 2– Searching post-buckling paths in the neighborhood of the critical point

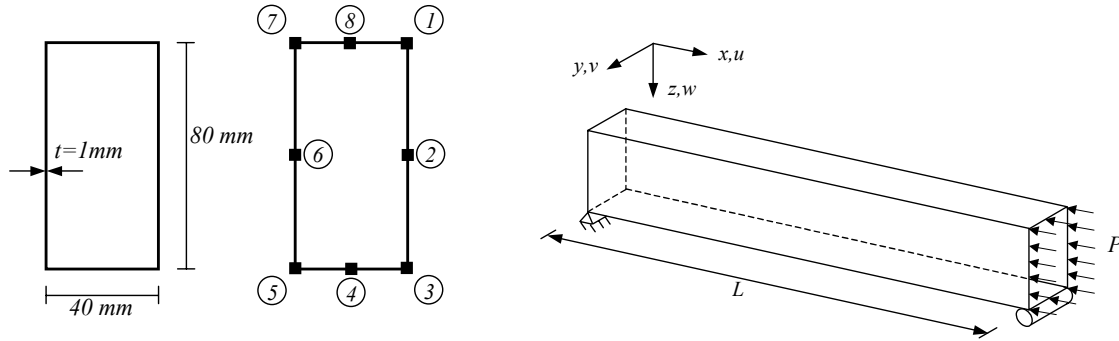
Example: the analysis of a simply supported RHS column

The numerical techniques shown above were employed to study the post-buckling behavior of the RHS presented in Fig. 3 in the context of the extended GBT formulation [2], with $L=250$ mm. The lowest critical load for this problem is $P_{CR}=35.42$ kN and the buckling mode is related to the local plate buckling of the webs – walls between nodes 1 and 3. The post-buckling paths are shown in Fig. 4, in a load-displacement representation, presenting the longitudinal displacement of node 1 (associated mainly with a passive coordinate) and the horizontal displacement of node 2 (associated mainly with a active coordinate). For the adopted discretization, which considered polynomials with up to 4 half-waves per mode, coordinate 31, related to local plate symmetrical web buckling mode, was the most active coordinate in the critical state, with a participation factor of 95.6%. Fig. 5 shows the member’s deformed configuration for $P/P_{CR}=2.54$.

Conclusions

The paper presented a numerical scheme to find post-buckling equilibrium paths for the analysis of perfect structural systems. It is just based on the fact that the sought equilibrium points fall outside the FP, and the only assumption is that the PBP, in the hyper-plane ${}^j a$ - P of the most active coordinate at the bifurcational state, moves away

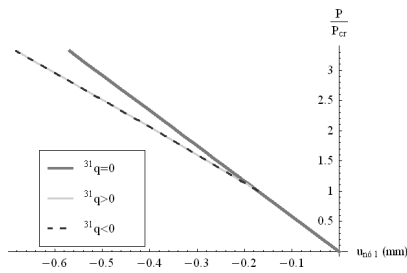
from the FP, independently of the values of the remaining coordinates and the loading parameter, which can take reversible values.



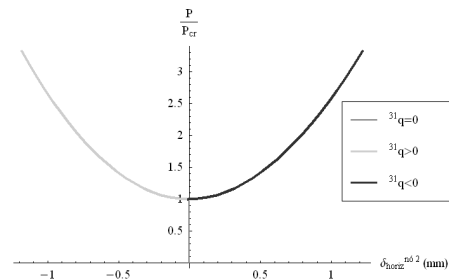
a) cross section properties and nodal discretization

b) member overview

Fig. 3 – The analyzed RHS



a) longitudinal displacement of node 1 at $x=250$ mm



b) horizontal displacement in the cross section plane for node 2 at $x=93.75$ mm

Fig. 4 – Displacements in the post-buckling range

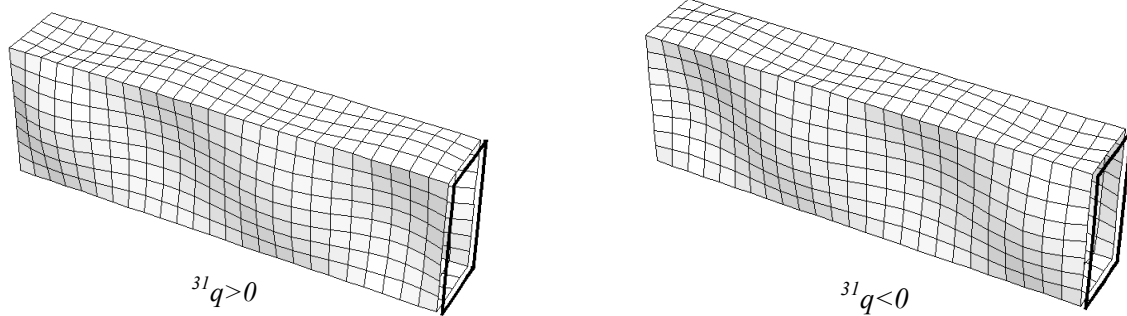


Fig. 5 – Member's configuration for $P/P_{cr}=2.54$

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