Non Normal Growth of Counter–Propagating Rossby Waves in Shear Instability

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Summary

Generalized stability analysis of the discrete spectrum optimal dynamics is presented and rationalized in terms of the interaction between two Counter Propagating Rossby Waves (CRWs), for conservative plane parallel shear flow. The singular vector decomposition which yields the optimal evolution is obtained in terms of the CRW interaction coefficient and their intrinsic phase speeds. The analysis is exemplified, for simplicity, on the Rayleigh model.

Introduction

The concept of Counter–Propagating Rossby Waves (CRWs) has been developed by [1] to explain normal mode instability in a two layers baroclinic model. [1] explained the instability in terms of two edge waves which propagate counter the mean zonal velocity via the Rossby wave mechanism of propagation. Each wave by itself is neutral, however the waves interact by inducing meridional velocity which advects the mean potential vorticity (PV) on the opposed CRW layer. According to the phase difference between the two CRWs the waves affect each other's propagation speed and growth. Normal mode instability is then achieved when the two CRWs are phase locked to propagate together in a growing configuration.

[2] generalized the CRW description in order to be applied on a general plane parallel shear flow which is linearly unstable and conserves PV. [2] showed that the generalized CRW equations become the Hamilton equations where the Hamiltonian, generalized momenta and coordinates are the eddy pseudo–energy, the CRW pseudo–momenta and phases, respectively. The generalization also rationalized the necessary condition for instability of Rayleigh [3] and Fjørtoft [4].

On a different path, [5] investigated the optimal non normal growth of eddies in both baroclinic and barotropic shear flows and developed the generalized stability theory (GST) of linear dynamical systems. [6] showed how a Singular Value Decomposition (SVD) of the propagator matrix of the dynamical system provides the optimal evolution which extracts maximal growth, in a given target time.

Our goal in this paper is to relate the eddy optimal growth in shear flow to the CRW dynamics. We analyze the discrete spectrum optimal dynamics from the CRW perspective and exemplify it on the barotropic Rayleigh model of shear instability [3]. Based on this understanding we suggest a scheme which extands the CRW description to the non modal growth in more general plane parallel shear flows.

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CRW Formulation in the Rayleigh Model

Consider the 2–D barotropic, inviscid and incompressible Rayleigh model [3], whose been discussed by [6], which its zonal basic state velocity and vorticity profiles are:

$$\overline{U}=\begin{cases}\Lambda b & \text{for } y\geq b\\ \Lambda y & \text{for } -b<y<b\\ -\Lambda b & \text{for } y\leq -b\end{cases}\qquad \text{and} \qquad \overline{q}=\begin{cases}0 & \text{for } y\geq b\\ -\Lambda & \text{for } -b<y<b\\ 0 & \text{for } y\leq -b\end{cases}\quad (1a,b)$$

Linearizion of the vorticity equation $(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla)q=0,$[2] then yields

$$(\frac{\partial}{\partial t}+\overline{U}\frac{\partial}{\partial x})q'=-v'\frac{\partial\overline{q}}{\partial y}, \qquad \text{where} \qquad \frac{\partial\overline{q}}{\partial y}=\Lambda[\delta(y-b)-\delta(y+b)]. \qquad (2a,b)$$

The discrete spectrum solution can be written then in terms of two CRW edge waves whose vorticity and streamfunctions are given by:

$$q'=[q_1(k,t)\delta(y+b)+q_2(k,t)\delta(y-b)]e^{ikx}, \quad \psi'=-\frac{1}{2k}\left[q_1(k,t)e^{-k|y+b|}+q_2(k,t)e^{-k|y-b|}\right]e^{ikx}(3a,b)$$

Writing $q_1=Q_1e^{i\varepsilon_1}$, $q_2=Q_2e^{i\varepsilon_2}$, and plug (3) in (2) we obtain the CRW equations:

$$\dot{Q}_1=\sigma Q_2\sin\varepsilon, \qquad \dot{Q}_2=\sigma Q_1\sin\varepsilon, \qquad (4a,b)$$

$$\dot{\varepsilon}_1=-kc_1^1-\sigma\frac{Q_2}{Q_1}\cos\varepsilon, \qquad \dot{\varepsilon}_2=-kc_2^2+\sigma\frac{Q_1}{Q_2}\cos\varepsilon, \qquad (4c,d)$$

when the CRW interaction coefficient and their two intrinsic phase speeds are :

$$\sigma=\frac{\Lambda}{2}e^{-K}, \quad c_1^1=\overline{U}(-b)\left(1-\frac{1}{K}\right), \quad c_2^2=\overline{U}(b)\left(1-\frac{1}{K}\right), \qquad (5a,b,c)$$

The CRW phase difference $\varepsilon=\varepsilon_2-\varepsilon_1$ and $K=2bk$ is the normalized wavenumber. (4) describes the interaction between two CRWs which interact by advecting the basic state vorticity of the opposed edge by the meridional velocity they induce there. If the CRWs are in phase they help each other to propagate counter the basic state wind while if they are anti

[2] The velocity vector $\mathbf{v}=(u,v)=(\overline{U}+u',v')$. The vorticity is taken as the scalar value in the plane's perpendicular direction: $q=\nabla\times\mathbf{v}=-\frac{\partial\overline{U}}{\partial y}+(\frac{\partial v'}{\partial x}-\frac{\partial v'}{\partial y})$. The basic state and the small perturbation are indicated by overbar and prime, respectively. We also use incompressibility $(\nabla\cdot\mathbf{v}=0)$ to write the perturbation in terms of the streamfunction ψ'; $(u',v')=(-\frac{\partial\psi'}{\partial y},\frac{\partial\psi'}{\partial x})$ and $q'=\nabla^2\psi'$.

phased they hinder the counter propagation rate. If, on the other hand, the northern CRW is shifted $\pi/2$ out of phase to the west of the southern wave the CRWs increase each other's amplitude where if the phase is $-\pi/2$ the CRWs decay each other. Growing normal modes are obtained when the two CRWs are phase locked each other to propagate together ($\dot{\varepsilon} = 0$) and have equal growth rates ($\dot{Q}_1/Q_1 = \dot{Q}_2/Q_2$). (4) can be also written in the matrix form:

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q}, \quad \mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad \mathbf{A} = -i \begin{pmatrix} kc_1^1 & \sigma \\ -\sigma & kc_1^1 \end{pmatrix} \tag{6a,b,c}$$

Hence, (6) enables the application of the GST of [5] on the CRW dynamics, in the enstrophy norm, as is shown next.

Generalized Stability Analysis of CRWs

The solution to (6) can be written in terms of the eigen decomposition of the propagator matrix $e^{\mathbf{A}t}$ or, alternatively, in terms of its Singular Value Decomposition: [3]

$$\mathbf{q}(t) = e^{\mathbf{A}t}\mathbf{q}(0) = (\mathbf{P}e^{\mathbf{L}t}\mathbf{R}^\dagger)\mathbf{q}(0) = (\mathbf{U}\Sigma\mathbf{V}^\dagger)\mathbf{q}(0) \tag{7}$$

The matrix $\mathbf{A}$ can be shown to be normal either when the CRWs have equal intrinsic phase speeds ($c_1^1 = c_2^2$), or in the limit of zero CRW interaction ($\sigma = 0$). The CRW interpretation for these cases is straightforward; when the CRWs have equal intrinsic phase speeds they should not help or hinder each other's propagation to remain phase locked and therefore maintain a phase difference of $\pi/2$ which is the optimal configuration for growth, as indicated by (4a,b). Hence, the CRWs would not gain any additional growth by moving relative to each other and the maximal growth is achieved by the eigenvectors themselves. With zero CRW interaction no growth is available and the two CRWs become two decoupled neutral edge waves which propagate with their own intrinsic phase speeds. [5] showed also that the maximum instantaneous growth rate is equal to the maximum eigenvalue of $(\mathbf{A} + \mathbf{A}^\dagger)/2$ which is simply σ in our case. The CRWs are then symmetric in amplitudes ($Q_1 = Q_2 = Q$) with a phase difference of $\varepsilon = \pi/2$. For a finite target time (4a,b) indicates that synchronous growth yields maximal growth and therefore (4a,b) become:

$$Q(t) = Q(0)\exp\left[\sigma \int_{t=0}^{t} \sin\varepsilon(t)\,dt\right]. \tag{8}$$

[3]$\mathbf{P}$ and $\mathbf{R}$ are the eigenvector and biorthogonal eigen vector matrices. $\mathbf{L}$ is the diagonal matrice of eigenvalues λ_j, $\mathbf{U}$ and $\mathbf{V}$ are Unitarian matrices and Σ is a diagonal matrix which composes real positive values ordered by magnitude along the diagonal. Only if the matrix $\mathbf{A}$ is Hermitian the two decompositions are identical. If the matrix $\mathbf{A}$ is normal (commutes with its Hermitian transpose; $\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger\mathbf{A}$), then it is a necessary and sufficient condition for its eigenvectors $\mathbf{p}_j$ to be orthogonal and $e^{Re(\lambda_j)t} = \sigma_j$. If however $\mathbf{A}$ is not normal then $\sigma_1 > e^{Re(\lambda_1)t}$ and can be shown to be the largest growth that system (6) can achieve in a given target time t. Hence, in order to obtain the optimal growth, $\mathbf{q}(0)$ should be chosen to be the first column unit vector of $\mathbf{V}$ since it would grow by σ_1 and be projected onto the first column of $\mathbf{U}$.

Thus the CRWs' optimal phase difference should cross $\varepsilon = \pi/2$, while optimally growing and since growth is symmetric with respect to $\varepsilon = \pi/2$ the initial and final optimal phases $(\varepsilon_0, \varepsilon_t)$, should satisfy $(\varepsilon_0 + \varepsilon_t)/2 = \pi/2$.

For wavenumbers whose normal modes are unstable it can be shown that the SVD becomes:

$$\mathbf{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -ie^{i\varepsilon_0} \\ -e^{-i\varepsilon_0} & -i \end{pmatrix}; \quad \mathbf{V} = \frac{1}{\sqrt{2}} \begin{pmatrix} ie^{-i\varepsilon_0} & 1 \\ i & -e^{i\varepsilon_0} \end{pmatrix}, \tag{9}$$

and

$$\Sigma = \Theta e^{\mathbf{L}t}; \quad \Theta = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}; \quad \theta = \left[\frac{\sin\left(\frac{\varepsilon_0+\varepsilon_+}{2}\right)}{\cos\left(\frac{\varepsilon_0-\varepsilon_+}{2}\right)} \right] \tag{10a,b,c}$$

where ε_+ indicates the growing normal mode phase difference. When $c_1^1 > c_2^2$, the CRWs must hinder themelves in order to be phase locked and thus the growing normal mode phase difference satisfies $\pi/2 < \varepsilon_+ < \pi$. Hence, any initial phase $\varepsilon_0 < \varepsilon_+$, increases with time, and since the optimal evolution must cross $\pi/2$, this yields $\varepsilon_0 < \pi/2 < \varepsilon_t$. From the same considerations, since $0 < \varepsilon_+ < \pi/2$ for $c_1^1 < c_2^2$, then $\varepsilon_t < \pi/2 < \varepsilon_0$. For target time infinity it is clear that the optimal perturbation would be eventually projected onto the most unstable mode $\mathbf{p}_1$ however this does not mean that the optimal way to excite the most unstable mode is to locate $\mathbf{q}(0)$ on this mode. Choosing $\mathbf{q}(0)$ to be in the direction of the biorthogonal vector $\mathbf{r}_1$, of the most unstable mode then the eigenvalue decomposition of (7) suggests that at time infinity $\mathbf{q}(t) \to |\mathbf{r}_1| e^{\Re(\lambda_1)t} \mathbf{p}_1$, where $|\mathbf{r}_1| > 1$ if $\mathbf{A}$ is not normal. Hence the biorthogonal phase difference should satisfy $\varepsilon_b = \pi - \varepsilon_+$.

Where only neutral normal modes exist transient optimal growth can still be achieved. The singular vectors in this case are the same as in (9) but the singular value matrix becomes

$$\Sigma = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}; \quad g = \sqrt{\frac{r - \cos\varepsilon_0}{r + \cos\varepsilon_0}}; \quad r = \frac{k|c_1^1 - c_2^2|}{2\sigma} \tag{11a,b,c}$$

For a given r the maximal possible growth G, denoted as the global optimal, is obtained when $\varepsilon_0 = \pi$,

$$G = \sqrt{\frac{r+1}{r-1}}, \qquad \text{at times} \qquad T = (n + \frac{1}{2})\frac{\pi}{\delta}, \qquad n = 0, 1, 2... \tag{12a,b,c}$$

and $\delta = \sigma\sqrt{r^2 - 1}$. The optimal evolution is synchronous with $(\varepsilon_0 + \varepsilon_t)/2 = \pm\pi/2$. The global optimal is being achieved when $(\varepsilon_0, \varepsilon_T) = (\pi, 0)$.

For synchronous growth in the neutral regime and $r > 1$, (4c,d) can be written as

$$\dot{\varepsilon} = 2\sigma(\cos\varepsilon - r) \tag{13}$$

Hence, the CRWs can never help each other enough to counter propagate against the shear which is always forcing the phase difference between the CRWs to decrease with time. Therefore, in order to obtain optimal growth the CRWs should cross $\varepsilon = \pi/2$ while $\varepsilon_0 > \varepsilon_t$. As in the case where $r < 1$ (of unstable normal modes), for very small target time $(\varepsilon_0, \varepsilon_t)$ are at the close vicinity of $\pi/2$, where as the target time increases they depart and the CRWs gain growth which increases until reaching the global optimal configuration $(\varepsilon_0, \varepsilon_t) = (\pi, 0)$. As the target time becomes slightly larger than the first global optimal the CRWs must begin with a slight decaying configuration (tilted with the shear), however still gaining growth while passing between $\varepsilon \in (\pi, 0)$. Since the motion is symmetric with repsect to $\pi/2$ the optimal structure will end up in a slightly decaying configuration and therefore optimal growth is smaller than the global optimal. As target time increases further, the CRWs must begin (and end) in a more and more decaying configuration and eventually for target time π/δ, the CRWs would start at phase difference of $\varepsilon(0) = -\pi/2$, will experience decay till they reach $\varepsilon = -\pi = \pi$, then will start growing until $\varepsilon = 0$ and finally decay again till $\varepsilon(t) = -\pi/2$. In this case the amount of decay exactly cancels the amount of growth and eventually no net growth is obtained. The next cycles, for target time between $(n\pi/\delta, (n+1)\pi/\delta)$ are identical to the first cycle $(0, \pi/\delta)$, except for the fact that the CRWs completed already $(n-1)$ full cycles which have yielded zero net growth.

Application to a General Shear Profile

Consider a general inviscid, incompressible, shear profile $\overline{U}(y)$ with a mean vorticity profile $\overline{q}(y)$. Then, writing all perturbation variables in the zonal Fourier form of $\eta(x,y,t) = \int_0^\infty \hat{\eta}(y,t,k)e^{ikx}dk$, we introduce the vorticity perturbation as

$$\hat{q}(y,t,k) = \int_{y'=-\infty}^{\infty} \left[\hat{q}(y',t,k)\delta(y'-y)\right] dy' \equiv \int_{y'=-\infty}^{\infty} \tilde{q}(y',t,k)dy' \,. \tag{14}$$

The "vorticity density kernel" $\tilde{q}$ induces a density streamfunction $\tilde{\psi}(y,y',t,k)$ which must satisfy $\tilde{q} = -k^2\tilde{\psi} + \tilde{\psi}_{yy}$ and therefore, $\tilde{\psi}(y,y',t,k) = \hat{q}(y,t,k)G(y,y')$ with the Green function $G(y,y') = -e^{-k|y-y'|}/2k$. Thus, the inversion of (14) can be written as

$$\hat{\psi}(y,t,k) = \int_{y'=-\infty}^{\infty} \hat{q}(y',t,k)G(y,y')dy'. \tag{15}$$

Substitute (14) and (15) in the linearized vorticity equation (2a) and write the vorticity in terms of amplitude and phase : $\hat{q}(y,t) = Q(y,t)e^{i\varepsilon(y,t)}$, we obtain for the real and the imaginary parts:

$$\dot{Q}(y)/Q(y) = -k\overline{q}_y(y)\int_{y'=-\infty}^{\infty} \left[Q(y')/Q(y)\right] G(y,y')\sin\varepsilon(y,y')dy' \,, \tag{16a}$$

$$\dot{\varepsilon}(y) = -k\left\{\overline{U}(y) + \overline{q}_y(y)\int_{y'=-\infty}^{\infty}\left[Q(y')/Q(y)\right]G(y,y')\cos\varepsilon(y,y')dy'\right\}, \qquad (16b)$$

where $\varepsilon(y,y') \equiv \varepsilon(y) - \varepsilon(y')$. (16) is the continuous analogue of (4) and can be interpreted as follows. Each CRW kernel changes its amplitude and phase due to meridional advection of the mean vorticity in its own layer, where the meridional wind is attributable to all other kernels and attenuated according to the Green function $G(y,y')$. The CRW kernel's amplitude $Q(y)$, grows due to all CRW kernels, located at $y' \neq y$, which are phase shifted by $0 < \varepsilon(y,y') < \pi$ and advects the mean vorticity in the opposite direction of $\overline{q}_y(y)$. The phase change is via the Rossby mechanism where the CRW kernels which are phase shifted by $-\pi/2 < \varepsilon(y,y') < \pi/2$ will "help" the CRW kernel at y to propagate counter the mean wind $\overline{U}(y)$, while the CRWs which are phase shifted by $\pi/2 < \varepsilon(y,y') < 3\pi/2$ will "hinder" the counter propagation. Hence the nature of interaction is the same as been exemplified on the two Rossby edge waves in the Rayleigh model, however now the interaction is in between infinite number of CRW kernels, were every kernel affects and being effected by all other kernels.

Finally, when (16) is being discretized in y to N layers it can be written as:

$$\dot{\hat{\mathbf{q}}} = \mathbf{A}\hat{\mathbf{q}}, \qquad \mathbf{A} = -ik\left[\mathbf{U} + \mathbf{Q_y}\mathbf{G}\right] \qquad (17a,b)$$

where $\mathbf{U}$, $\mathbf{Q_y}$ are diagonal matrices of the shear and the vorticity gradient. $\mathbf{G}$ is the Green function Hermitian matrice. GST analysis can be then applied on (17), while the simple nature of CRW interaction is preserved.

Reference

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