Approximate 3D Elastodynamic Greens Functions for an Inhomogeneous Elastic Medium

F. J. Sánchez-Sesma[1], L. Ramírez-Guzmán[2], F. Luzón[3]

Summary

For a constant-gradient elastic isotropic medium, i.e. a medium with linear variation of wave velocity, the rays from a point source are known to be circular. The associated wave fronts are circular cylinders or spheres in two or three-dimensions, respectively. This fact allowed us to construct analytical approximations for the corresponding Green's functions. In 2D the elementary impulsive sources are unit line forces per unit length (anti-plane *SH* line source and the in-plane *P–SV* line sources, respectively). In 3D the sources are vertical and horizontal unit point forces. The derivation of the analytical approximations starts with a generalization of the homogeneous Green's functions and relies on the asymptotic ray theory to establish the travel times and the geometrical spreading factors. Our approximation accounts for both near-source effects and low frequencies. The adequate behavior of our expressions in 3D is tested by comparing them with results from an explicit fourth order finite difference scheme.

Introduction

Geological and topographic local conditions can induce amplifications either in seismic ground motion or in the shaking duration or both. For scientific and practical reasons it is of interest to assess the significance of such increments. During the last decades site effects have been studied both theoretically and experimentally. In fact, there are regions where the data available do allow us to forecast, within a reasonable range, the dynamic characteristics of the soil, but when the coverage is not enough or the information from the seismograms do not cover all possible source scenarios, the only available tool is numerical simulation. As computational power grows and numerical methods improve our models are trustworthy. The models of San Bernardino [1] and Kanto Basin [2], just to mention some works, show large differences between two- and three-dimensional analyses. Besides, field prospecting tests show increments of the stiffness as the confinement grows, which is strongly related to depth. Therefore, even in a homogeneous soil we should consider a variation in the wave velocities as the depth increases. In some circumstances it is enough assume a set of plane layers. In this paper we extend the results of previous work in 2D [3] and give analytical expressions of the 3D Green's functions for an isotropic elastic space with linear variation in velocities.

[1] Instituto de Ingeniería, UNAM; Ciudad Universitaria, México DF; 04510 Mexico; sesma@servidor.unam.mx

[2] Laboratory of Computational Mechanics; Civil and Environmental Department; Carnegie Mellon University; Forbes 5000, Pittsburgh PA, 15217 USA, lramirez@andrew.cmu.edu

[3] Dpto. de Física Aplicada; Universidad de Almería; Cañada de San Urbano s/n Almería, 04120 Spain, fluzon@ual.es

Formulation of Problem and 3D Green´s Functions

The elastic domain of interest is defined by $z > -1/\gamma = -h$, in which wave velocities and density are given by

$$c(z) = c_0 \cdot \left(\frac{1+\gamma \cdot z}{1+\gamma \cdot z_0} \right) = c(0) \cdot (1+\gamma \cdot z), \tag{1}$$

$$\rho(z) = \rho_0 \cdot \left(\frac{1+\gamma \cdot z}{1+\gamma \cdot z_0} \right)^n = \rho(0) \cdot (1+\gamma \cdot z)^n, \tag{2}$$

where c = wave speed (α for P or β for S), ρ = density, $\alpha(-h) = \beta(-h) = 0$ and $n \geq 0$. The zero subscript refers to a property at source level. This is a constant-gradient elastic isotropic medium in which the rays from a point source are known to be circular. The associated wave fronts are either circular cylinders or spheres in two or three-dimensions, respectively. For a point source at (0, 0, z_0) the wave travel time to a point (x, y, z) is given by

$$\tau = \frac{h}{c(0)} \ln \left(\frac{R_2 + R_1}{R_2 - R_1} \right), \tag{3}$$

where $R_1 = \sqrt{r_h^{\ 2} + (z - z_0)^2}$, $R_2 = \sqrt{r_h^{\ 2} + (z + z_0 + 2h)^2}$, $r_h = \sqrt{(x - x_0)^2 + (y - y_0)^2}$, and $c(0)$ = wave speed at $z = 0$. The wave fronts are spheres of radius R_w, (see Fig. 1) whose equation is (see [4])

$$r_h^{\ 2} + \{z - [(z_0 + h)\cosh(\beta(0)\tau / h) - h]\}^2 = R_w^{\ 2}, \tag{4}$$

where $R_w = (z_0 + h)\sinh(\gamma\, c(0)\tau) = R_1 R_2 / 2(z+h)$ =radius of the wave front either for a known travel time or at a given point.

The expression for the displacement, in frequency is given by ray theory [5] as

$$u(\omega, x) = \left(\frac{\rho(x_r)c(x_r)}{\rho(x)c(x)} \right)^{\frac{1}{2}} \aleph(x) \cdot u(\omega, x_r) e^{-i\omega(t-\tau)}, \tag{5}$$

where ω = angular frequency, τ =travel time, $\boldsymbol{x}_r$ = reference position in the ray considered and $\aleph$ ($\boldsymbol{x}$)= geometrical spreading factor. The displacement $\boldsymbol{u}$ and the velocity c represent both P or S waves.

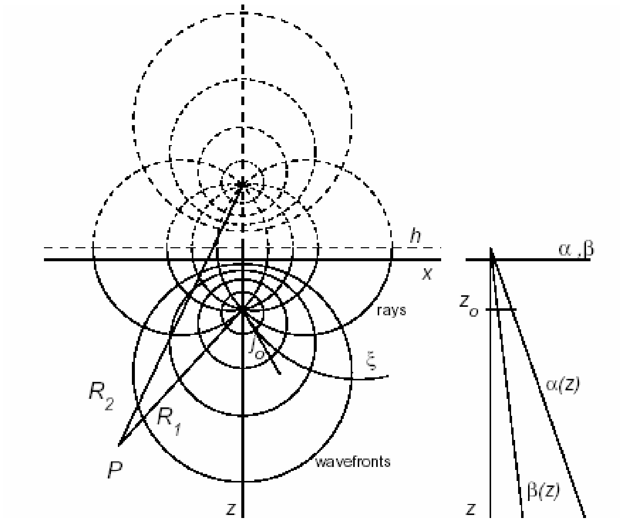

Figure 1.- *Medium considered. Drawn the wave fronts and rays.*

Cerveny and Psencik [6] gave the methodology to get $\aleph(\boldsymbol{x})$. Following their equations (5.5), (5.23), (5.25) and (6.5) of that paper, we obtain

$$\aleph(x) = \frac{c_0 \tau_r}{R_w} \left(\frac{1+\gamma z_0}{1+\gamma z} \right), \tag{6}$$

for a point source. Thus, considering the Eqs. (1) to (2), (5) and (6) we obtain

$$u(\omega, x) = \left(\frac{1+\gamma z_0}{1+\gamma z} \right)^{\frac{n+3}{2}} \frac{c_0 \tau_r}{R_w} u(\omega, x_r) e^{-i\omega\tau_\alpha} . \tag{7}$$

To get the approximate fundamental solutions for a point source we follow a similar procedure as described in [3]. We know that the exact radial and tangential displacements for a unit point force in a homogeneous full space are

$$u_r = \frac{1}{4\pi\mu} \left(\frac{1}{r} \right) f_1 \cos\theta , \quad \text{and} \quad u_\theta = -\frac{1}{4\pi\mu} \left(\frac{1}{r} \right) f_2 \sin\theta \tag{8}$$

where the Stokes's functions are

$$f_1 = \left(\frac{\beta}{\alpha} \right)^2 \left[1 - \frac{2i}{\omega\tau_\alpha} - \frac{2}{(\omega\tau_\alpha)^2} \right] e^{-i\omega\tau_\alpha} + \left[\frac{2i}{\omega\tau_\beta} - \frac{2}{(\omega\tau_\beta)^2} \right] e^{-i\omega\tau_\beta} , \tag{9}$$

$$f_2 = \left(\frac{\beta}{\alpha} \right)^2 \left[\frac{i}{\omega\tau_\alpha} + \frac{1}{(\omega\tau_\alpha)^2} \right] e^{-i\omega\tau_\alpha} + \left[1 - \frac{2i}{\omega\tau_\beta} - \frac{2}{(\omega\tau_\beta)^2} \right] e^{-i\omega\tau_\beta} . \tag{10}$$

In these equations r is the distance between the receiver and the source, θ is the angle between the directions of the force and the source-receiver line and τ_α, τ_β are the travel times for P and S waves. We propose a generalization of the displacement due to a unit vertical point force as

$$G_{R_w z} = \Lambda \frac{1}{4\pi\mu_0}\left(\frac{1}{R_w}\right) f_1 \cos\theta_0 \,, \text{ and } \quad G_{j_0 z} = -\Lambda \frac{1}{4\pi\mu_0}\left(\frac{1}{R_w}\right) f_2 \sin\theta_0 \qquad (11)$$

where $\Lambda = \left(\frac{1+\gamma z_0}{1+\gamma z}\right)^{\frac{n+3}{2}}$, and θ_0 is the take-off angle. Our proposal applies to the 'radial' and 'angular' components which are the local normal and tangents to wave front.

The expressions for a horizontal force along the x axis are

$$G_{R_w x} = \Lambda \frac{1}{4\pi\mu_0}\left(\frac{1}{R_w}\right) f_1 \sin\theta_0 \cos\phi \,, \qquad (12)$$

$$G_{j_0 x} = \Lambda \frac{1}{4\pi\mu_0}\left(\frac{1}{R_w}\right) f_2 \cos\theta_0 \cos\phi \,, \qquad (13)$$

$$G_{\phi x} = -\Lambda \frac{1}{4\pi\mu_0}\left(\frac{1}{R_w}\right) f_2 \sin\phi \,. \qquad (14)$$

The expressions are correct for high frequency and far field. An inspection of (11)-(14) shows that our proposal also gives the exact solution for near-source field. In order to give general expressions in Cartesian coordinates for the approximate Green's function let's define the longitudinal vector $l_i\,(\theta,\phi)$, which is tangent to the ray and forms at each point along the ray an angle θ with z axis (ϕ is the azimuthal angle, constant along one given ray, $\cos\phi = (x-x_0)/r_h$ and $\sin\phi = (y-y_0)/r_h$), and the transverse components $h_i(\phi)$ and $v_i(\theta,\phi)$, polarized in the horizontal and vertical planes, respectively. They are

I	$l_i(\theta,\phi)$	$h_i(\phi)$	$v_i(\theta,\phi)$
1	$\sin\theta\cos\phi$	$-\sin\phi$	$\cos\theta\cos\phi$
2	$\sin\theta\sin\phi$	$\cos\phi$	$\cos\theta\sin\phi$
3	$\cos\theta$	0	$-\sin\theta$

Therefore, we can write

$$G_{ij} = \Lambda \frac{1}{4\pi\mu_0 R_w} \left\{ f_1 \, l_i(\theta,\phi) l_j(\theta_0,\phi) + f_2 \left[h_i(\phi) h_j(\phi) + v_i(\theta,\phi) v_j(\theta_0,\phi) \right] \right\}. \quad (15)$$

which exhibits the longitudinal and the transverse components. It also shows that asymptotically the Stokes' factors lead to the *P*, *SH* and *SV* components, respectively.

In order to verify our approximation we calculated several examples. We show one of them. In this model (Fig. 2) $\alpha(0) = 200 m/s$, $\beta(0) = 100 m/s$, $\rho(0) = 1 T/m^3$ and *h*= 1000 m. The stations are grouped in two sets (A y B) and their positions are shown in Fig. (2). The vertical force is a triangular pulse with $t_p=0.6$ and $t_s=0.31$ *s*. We plot our analytical results *versus* the numerical solutions obtained with a fourth order FDM using a mesh of 700 x 700 with intervals of 10 m. The comparison, presented in Fig. (3), shows that the agreement is excellent and that the proposed formulae give the right amplitude and travel time for both waves *P* and *S* waves.

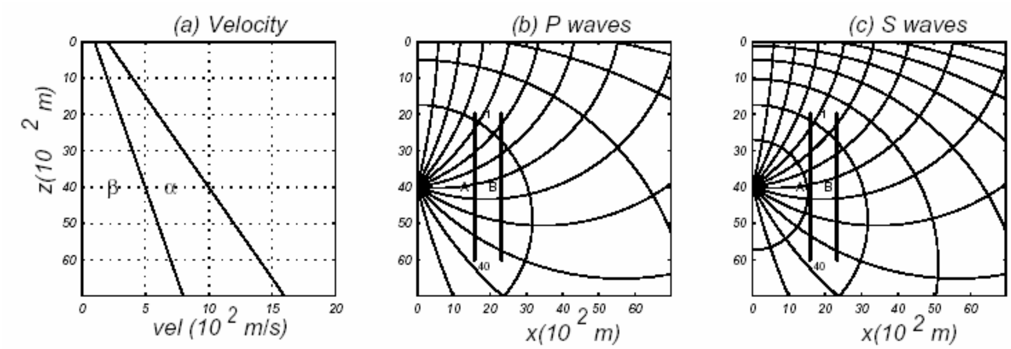

Figure 2.- *Properties in a heterogeneous medium a) velocities* $\alpha(0) = 200 m/s$, $\beta(0) = 100 m/s$, $\rho(0) = 1 T/m^3$ *and h= 1000 m, b) and c) rays and wave fronts every 3 s. Set of stations A and B in thick line.*

Conclusions

A 3D Green's function set was presented for an elastic isotropic and heterogeneous medium. We showed a good agreement of our expressions with a FDM calculation. The advantage of our anzats is its ease in structure and computation. The formulae are regular in all the domain and the singularities in the source can be handled with conventional methods. A definition of a norm to establish the range of validity is a matter of further scrutiny and will be discussed elsewhere. These Green's functions extend considerably the realm of BEM. In fact, with the same computational complexity as in a homogeneous case we can solve a heterogeneous medium.

Acknowledgements

Thanks are given to T. Furumura for his keen critical remarks. This research have had the generous support of DGAPA-UNAM, Mexico, under grant IN121202 and

CONACYT, Mexico, under project NC-204, by CICYT, Spain, under REN2002-04198-CO2-02/RIES, by the E.U. with FEDER, and the team RNM-194 of Junta de Andalucía.

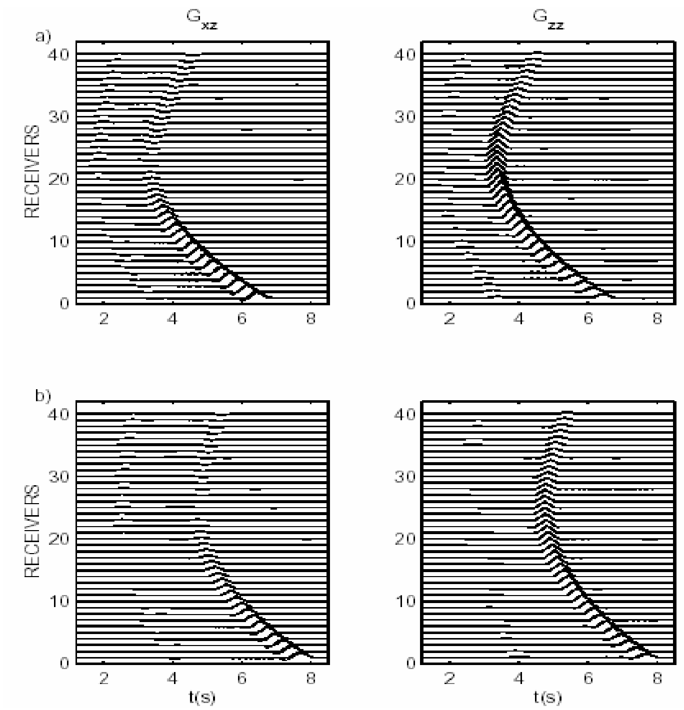

Figure 3.- a) and b) *Displacements in a heterogeneous medium for a vertical force in the receptors sets A and B. Material properties and positions of both source and receivers are shown in Fig. 2. In continuous line the exact analytical solution and in dotted line the solution obtained with a fourth order FDM.*

References

1 Frankel, A., (1993): "Three-dimensional simulations of ground motion in the San Bernardino Valley, California, for hypothetical earthquakes on the San Andreas fault", *Bull. Seism. Soc. Am.,* 83, 1020-1041.

2 Hisada, Y., K. Aki & T. Teng, (1993): "3D Simulations of Surface Wave Propagation in the Kanto Sedimentary Basin, Japan, Part 2: Application of the Surface Wave BEM", *Bull. Seism. Soc. Am.* , 83, 1700-1720.

3 Sánchez-Sesma, F. J., R. Madariaga & K. Irikura, (2001): "An approximate elastic 2D Green's functions for a constant gradient medium", *Geophys. J. Int*., 146, 237-248.

4 Ben-Menahem, A. W.B. Beydoun, (1985): "Range of validity of seismic ray and methods in general inhomogeneous media-I General theory", *Geophys. J. R. ast. Soc*., 82, 207-234.

5 Cerveny V. & Ravindra R, (1971): *Theory of seismic head waves*, University Toronto Press, Toronto.

6 Cerveny V. & I. Psencik, (1979): "Ray Amplitudes of Seismic Body Waves in Laterally Inhomogeneous Media", *Geophys. J. R. Astr. Soc*., 57, 91-106.