2.5D Transient Heat Transfer (Conduction and Convection)

Using the BEM

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Summary

This paper summarizes the study of three-dimensional transient heat transfer through cylindrical structures with irregular cross-sections, bounded by a homogeneous elastic medium. This system is subjected to heat generated by spherical or cylindrical convective sources placed somewhere in the media. The Boundary Element Method (BEM) is used to formulate the problem, handling both the conduction and the convection phenomena.

This model is used to compute the transient temperature field through a cylindrical square element containing a cylindrical circular inclusion, inserted in a homogeneous infinite medium. The final equations are defined imposing both temperatures and heat flux continuity along the boundary interfaces between the different media. The calculations are first performed in frequency domain for a wide range of frequencies and axial wavenumbers. Time series responses are then obtained by means of (fast) inverse Fourier transforms into space-time. The appropriate fundamental solution (Green's functions) employed in this BEM model incorporates a variable related to the convection phenomenon.

Introduction

Most heat transfer problems involve heterogeneous elements and unsteady changes of energy between different media. Therefore, the formulations for studying those systems should contemplate the transient heat phenomena. Several numerical approaches have been developed to study the heat transfer, such as the finite elements (e.g. Bathe [1]), the finite differences and also the boundary elements method (Brebbia et al. [2]). Of these techniques the BEM is the only one that requires the discretization of the material boundaries alone, which makes the description of the region quite compact.

Most of the known schemes devised to solve transient diffusion heat problems have been formulated in the time domain ("time-marching" approach) or else use Laplace transforms. The BEM has been employed in both techniques. In the "time-marching" approach, the BEM can be used to compute the solution directly in the time domain, step by step, at successive time increments. Chang et al. [3] presented the first time-domain direct boundary integral method to study planar transient heat conduction. A disadvantage of the "time-marching" techniques is that they can exhibit unstable solutions. An alternative is to apply a Laplace transform to move the solution from the

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time domain to a transformed variable. Rizzo and Shippy [4] used a Laplace transform associated with a boundary integral representation for transient heat conduction analysis.

One drawback of using Laplace transforms is the loss of accuracy in the inversion process, which amplifies small truncation errors. Stehfest [5] has proposed a more stable algorithm to overcome this problem. A time Fourier transform approach has been suggested by the authors (Tadeu et al. [6]) to deal with the time variable in the transient heat conduction equation.

In the present work, the time Fourier Transform is again used to compute the transient heat transfer when cylindrical inclusions of infinite length are located inside a homogeneous elastic medium. However, the convection phenomenon also takes part in this formulation.

Three-dimensional problem formulation

The transient heat transfer involving the conduction and convection phenomena, in the frequency domain, can be expressed by the diffusion equation below, when constant velocities, V_x, V_y and V_z, are assumed along the domain in the x, y and z directions,

$$\left(\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) - \frac{1}{K}\left(V_x \frac{\partial}{\partial x} + V_y \frac{\partial}{\partial y} + V_z \frac{\partial}{\partial z} \right) + \left(\sqrt{\frac{-i\omega}{K}} \right)^2 \right) \hat{T}(\omega, x, y, z) = 0, \quad (1)$$

in which $\hat{T}(\omega, x, y, z)$ is temperature, $K = k/(\rho c)$ is the thermal diffusivity, k is the thermal conductivity, ρ is the density and c is the specific heat, $i = \sqrt{-1}$ and ω is the frequency. The fundamental solution of eqn. (1) for a heat point source placed in an unbounded medium at $(0, 0, 0)$, can be given by

$$\hat{T}_f(\omega, x, y, z) = \frac{e^{\frac{V_x x + V_y y + V_z z}{2K}}}{2k\sqrt{x^2 + y^2 + z^2}} e^{-i\sqrt{\frac{V_x^2 + V_y^2 + V_z^2}{4K^2} - \frac{i\omega}{K}}\sqrt{x^2 + y^2 + z^2}}. \quad (2)$$

When the geometry of the problem does not change along one direction (z) the full 3D problem can be expressed as a summation of simpler 2D solutions. This requires the application of a Fourier transformation along that direction, which can be expressed as a summation of 2D solutions with different spatial wavenumbers k_z (Tadeu and Kausel[7]).

The application of a spatial Fourier transformation to $\dfrac{e^{-i\sqrt{\frac{V_x^2 + V_y^2 + V_z^2}{4K^2} - \frac{i\omega}{K}}\sqrt{x^2 + y^2 + z^2}}}{\sqrt{x^2 + y^2 + z^2}}$ along the z

direction, leads to

$$\tilde{T}_f(\omega, x, y, k_z) = \frac{-i\, e^{\frac{V_x x + V_y y + V_z z}{2K}}}{4k} H_0\left(\sqrt{-\frac{V_x^2 + V_y^2 + V_z^2}{4K^2} - \frac{i\omega}{K} - (k_z)^2}\; r_0 \right), \quad (3)$$

where $H_0(\ )$ are second kind Hankel functions of the order 0, and $r_0 = \sqrt{x^2 + y^2}$. The full three-dimensional heat field is then achieved by applying an inverse Fourier transform along the k_z domain. If we assume the existence of virtual sources, equally spaced sufficiently far apart, distance L_z (along z), to prevent spatial contamination (Bouchon and Aki [8]), this inverse Fourier transformation can be formulated as a discrete summation, which enables the solution to be obtained by solving a limited number of two-dimensional problems,

$$\hat{T}(\omega, x, y, z) = \frac{2\pi}{L} \sum_{m=-M}^{M} \tilde{T}(\omega, x, y, k_{zm}) e^{-ik_{zm}z}, \tag{4}$$

with axial wavenumber, k_{zm}, being given by $k_{zm} = \frac{2\pi}{L_z} m$.

Boundary Element Method formulation

In a homogeneous isotropic solid medium of infinite extent, containing an inclusion of volume V, bounded by a surface S, and subjected to an incident heat source given by $\tilde{T}^{inc}$, the reciprocity theorem can be applied to derive the relevant boundary integral equations,

$$\bar{c}\,\tilde{T}^{(ext)}(x_0, y_0, k_z, \omega) = \int_S q^{(ext)}(x, y, \nu_n, k_z, \omega) G^{(ext)}(x, y, x_0, y_0, k_z, \omega)\,ds$$
$$- \int_S H^{(ext)}(x, y, \nu_n, x_0, y_0, k_z, \omega)\tilde{T}^{(ext)}(x, y, k_z, \omega)\,ds \tag{5}$$
$$- \int_S G^{(ext)}(x, y, x_0, y_0, k_z, \omega)\tilde{T}^{(ext)}(x, y, k_z, \omega)\,\mathrm{V}_n^{(ext)}ds + \tilde{T}_{inc}(x_0, y_0, k_z, \omega)$$

$$\bar{c}\,\tilde{T}^{(int)}(x_0, y_0, k_z, \omega) = \int_S q^{(int)}(x, y, \nu_n, k_z, \omega) G^{(int)}(x, y, x_0, y_0, k_z, \omega)\,ds$$
$$- \int_S H^{(int)}(x, y, \nu_n, x_0, y_0, k_z, \omega)\tilde{T}^{(int)}(x, y, k_z, \omega)\,ds \tag{6}$$
$$- \int_S G^{(int)}(x, y, x_0, y_0, k_z, \omega)\tilde{T}^{(int)}(x, y, k_z, \omega)\mathrm{V}_n^{(int)}\,ds$$

In eqns (5) and (6), the interior and exterior domain are identified by the superscripts *int* and *ext* respectively, ν_n is the unit outward normal along the boundary, G and H are respectively the fundamental solutions (Green's functions) for the temperature ($\tilde{T}$) and heat flux (q), at (x, y) due to a virtual point heat load applied at (x_0, y_0), $\mathrm{V}_n = \mathrm{V}_x n_x + \mathrm{V}_y n_y$, $\bar{c}$ is a constant defined by the shape of the boundary, with a value of $1/2$ if $(x_0, y_0) \in S$ when the boundary is smooth.

The two-and-a-half dimensional Green's functions, for temperature in Cartesian coordinates, are those for an unbounded solid medium,

$$G(x,y,x_0,y_0,k_z,\omega) = \frac{-i}{4k} e^{\frac{V_x x + V_y y}{2K}} H_0 \left(\sqrt{\frac{-V_x^2 - V_y^2}{4K^2} + \frac{-i\omega}{K} - (k_z)^2} \; r_0 \right) \tag{7}$$

Applications

The BEM formulation has been used to study heat propagation in the presence of a square cylindrical solid inclusion (medium 2) buried in an unbounded solid medium (medium 1). A circular cylindrical inclusion (medium 3) is, in turn, modeled inside this square inclusion (see Fig. 1). The inclusions have been modeled using 500 straight boundary elements. This system is subjected to a heat point source, placed in the host medium at O ($x_0 = -0.15$ m, $y_0 = -0.1$ m, $z_0 = 0.0$ m), which starts emitting energy at $t \approx 0.76$ h. Its power is increased linearly from 0.0 W to 1000.0 W, reaching maximum power at $t \approx 3.46$ h. This value is maintained for a period of $t \approx 2.72$ h. The power is then reduced linearly to 0.0 W, which occurs at $t \approx 8.89$ h. The heat field is computed for several grid of receivers in the frequency range of $\left[0, 128e^{-5} \text{Hz} \right]$, with a frequency increment of $0.5e^{-5}$ Hz, giving a time period of 55.56 h.

The convection velocity (in the y direction) affecting the host medium is -1×10^{-6} m/s, while for the rectangle and circle inclusions the y velocities of 2×10^{-6} m/s and -6×10^{-6} m/s have been ascribed, respectively.

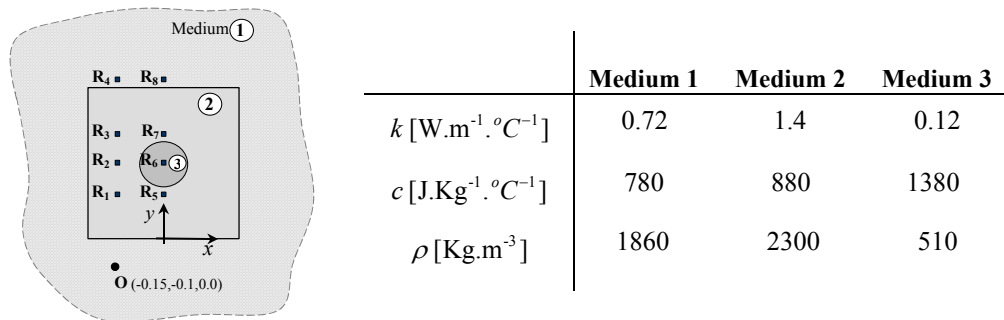|	Medium 1	Medium 2	Medium 3
k [W.m^{-1}.$^oC^{-1}$]	0.72	1.4	0.12
c [J.Kg^{-1}.$^oC^{-1}$]	780	880	1380
ρ [Kg.m^{-3}]	1860	2300	510

Figure 1. Geometry of the problem, position of a set of receivers at $z = 0.0$ m, and thermal properties of the material

In Figure 2 a set of contour plots at $t = 10$ h and $t = 20$ h displays the temperature field along a transversal grid of receivers placed at $z = 0.0$ m and also through a longitudinal grid of receivers, distributed in the $x z$ plane, crossing the two inclusions at $y = 0.25$ m.

At $t = 10$ h (Fig. 2a) the receivers behind the cylindrical circular inclusion (opposite the source side) register lower temperatures along the transversal grid of receivers than the other regions that are at the same distance from the source, which means that the heat flows through this inclusion more slowly. This behaviour is explained by the effect of the thermal properties (low diffusivity) of the circular inclusion and also by the presence of a negative convection velocity in this medium. As time elapses, the temperature differences between the region behind the inclusion and in its vicinity decrease ($t = 20$ h in Fig. 2a), confirming the energy equilibrium evolution.
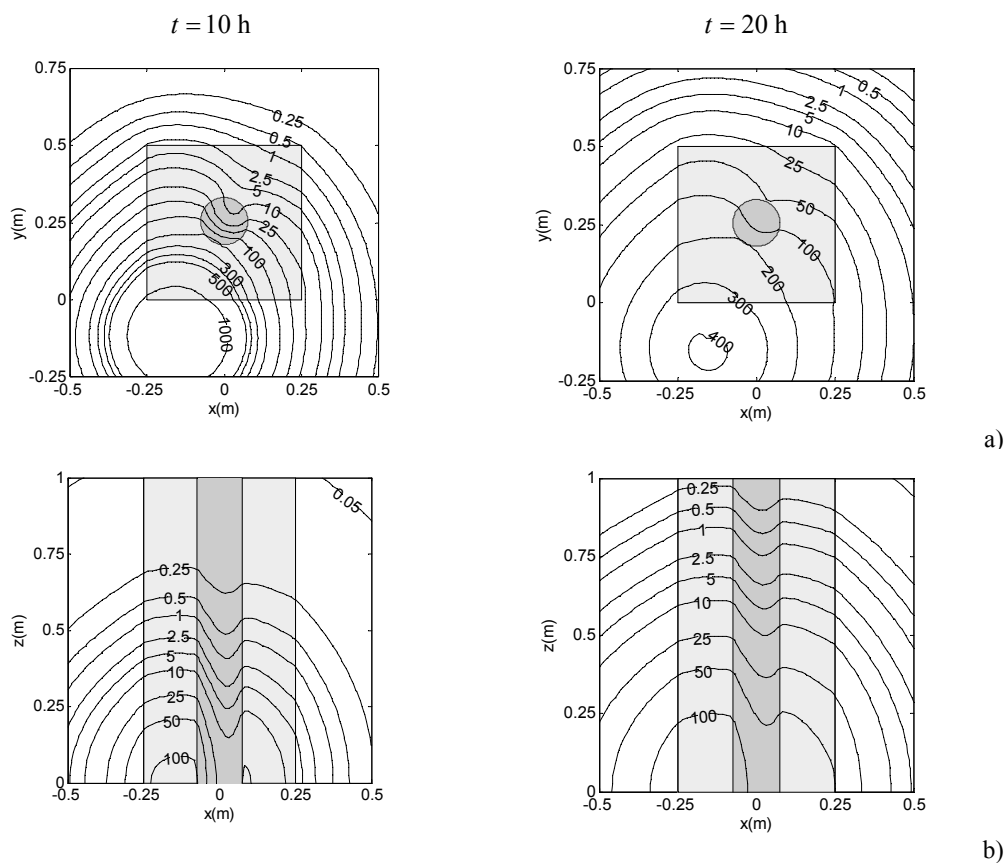

Figure 2. Temperature fields (ºC) registered at $t = 10$ h and $t = 20$ h : a) Transversal grid of receivers at $z = 0.0$ m ; b) Longitudinal grid of receivers at $y = 0.25$ m .

In contrast, the high thermal diffusivity of the medium 2, associated with its convection velocity, allows faster energy propagation through this domain. The temperature field along the longitudinal grid of receivers (Fig. 2b) shows lower

temperature amplitudes at the receivers placed inside the circular inclusion when compared with the neighbouring region. In the snapshots taken later ($t = 20\,\text{h}$), it is interesting to note that the temperature rises in the zones a long away from the source, since the energy is still flowing away through the domain, while the temperature closer to the heat source decreases sharply.

Conclusions

A BEM model has been implemented and used to compute three-dimensional transient heat transfer along a system of cylindrical inclusions of infinite length, located inside a homogeneous elastic medium and heated by a spherical heat load placed inside the host medium. The phenomena studied here include heat conduction and convection. The incorporation of convection diffusion changes the heat responses in agreement with the velocities assumed for each material.

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