Elastic-Plastic Modeling of Kinematic Hardening Materials Based on the Corotational Rate of the Logarithmic Strain Tensor

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Summary

In this paper, based on the additive decomposition of the corotational rate of the logarithmic strain tensor an elastic-plastic modeling of kinematic hardening materials is introduced. In this model, the elastic constitutive equation as well as the flow rule and hardening equation are expressed in terms of the corotational rate of the elastic and plastic parts of the logarithmic strain tensor. As an application, the simple shear problem is solved and the stress components are plotted versus shear displacement for a kinematic hardening material.

Introduction

In small deformations of elastic-plastic materials in the classical plasticity, the strain tensor is decomposed additively into elastic and plastic parts. If deformations are large, this decomposition is no longer valid [1]. Hence, in kinematical analysis of large deformation elastic-plastic materials, other types of additive decompositions as well as multiplicative decompositions are used. One of the most common additive decompositions in the large deformation analysis, which has been used by Nemat-Nasser [2], is the decomposition of the strain rate tensor into elastic and plastic parts. Recently, Naghdabadi et al. [3] have proposed the additive decomposition of the corotational rate of the logarithmic strain tensor into elastic and plastic parts. They used this decomposition for isotropic hardening materials. The multiplicative decompositions have also been widely used. The most common multiplicative method is the decomposition of deformation gradient, which had been used by Lee [4].

In the theory of plasticity, constitutive models consist a yield surface, a flow rule and an evolution equation. In finite deformation plasticity in order to describe the material response independent of rigid rotations, the constitutive model should be based on the corotational rates. A corotational rate of a tensor valued quantity, $\mathbf{T}$, represents the rate of change of such quantity with respect to a coordinate system, which rotates instantaneously with spin $\mathbf{\Omega}^{\circ}$ [5]:

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$$\mathbf{T}^{o} = \dot{\mathbf{T}} + \mathbf{T}\boldsymbol{\Omega}^{o} - \boldsymbol{\Omega}^{o}\mathbf{T} \tag{1}$$

where $\dot{\mathbf{T}}$ is the material time derivative of tensor **T** with respect to a fixed coordinate system. Different spin tensors have been defined in continuum mechanics, thus different corotational rates have been introduced. For example two well-known corotational rates are Green-Naghdi (Zaremba) and Jaumann rates [6], which are defined on the basis of spin tensors, $\boldsymbol{\Omega}$ and **W**, respectively. $\boldsymbol{\Omega}$ is the spin tensor associated with the proper orthogonal rotation tensor, **R** and **W** is the skew-symmetric part of the velocity gradient tensor. E-rate is another type of corotational rates. The spin tensor related to E-rate, $\boldsymbol{\Omega}^{E}$, is the Eulerian spin tensor [7]. Xiao et al. [8] introduced the logarithmic corotational rate, called log-rate and its associated spin tensor. They have used frequently this corotational rate in the large deformation elastic-plastic modeling of solids [9, 10].

In this paper, the kinematic decomposition of the corotational rate of the logarithmic strain tensor [3] is used and based on the logarithmic flow rule proposed by Naghdabadi et al. [11] for rigid plastic hardening materials, the plastic part of the corotational rate of the logarithmic strain tensor is related to the difference of the deviatoric Cauchy stress and back stress tensors. The kinematic hardening model relates the corotational rate of the back stress tensor to the plastic part of the corotational rate of the logarithmic strain tensor. As an application of the proposed elastic-plastic constitutive modelling, the simple shear problem is solved for different corotational rates and the stress components are plotted versus the shear displacement.

Kinematic and Constitutive Modeling

Here the corotational rate of the logarithmic strain tensor is decomposed additively into elastic and plastic parts, as following:

$$\mathbf{e}^{o} = \mathbf{e}^{oe} + \mathbf{e}^{op} \tag{2}$$

where the super script "o" denotes the corotational rate, also $\mathbf{e}^{oe}$ and $\mathbf{e}^{op}$ are elastic and plastic parts of the corotational rate of the logarithmic strain tensor, respectively. Through out this paper the superscripts "e" and "p" denote elastic and plastic parts of tensors, respectively. The elastic part of the corotational rate of the logarithmic strain tensor is defined through the following hypo-elastic constitutive equation:

$$\mathbf{e}^{oe} = \frac{1}{2G}\boldsymbol{\sigma}^{o} - \frac{\nu}{E}\mathrm{tr}(\dot{\boldsymbol{\sigma}})\mathbf{I} \tag{3}$$

where G, E, ν, $\boldsymbol{\sigma}$ and **I** are modulus of rigidity, Young's modulus, Poisson's ratio, Cauchy stress and identity tensor, respectively. The plastic part of the corotational rate of the logarithmic strain tensor, $\mathbf{e}^{op}$ is obtained from the logarithmic flow rule. The logarithmic flow rule, which has been proposed by Naghdabadi et al. [11] relates the plastic part of the corotational rate of the logarithmic strain tensor to the difference of the deviatoric Cauchy stress and back stress tensors:

$$\mathbf{e}^{op} = \dot{\phi}(\mathbf{S} - \boldsymbol{\alpha}) \tag{4}$$

where $\mathbf{S} = \boldsymbol{\sigma} - \mathrm{tr}(\boldsymbol{\sigma})\mathbf{I}/3$ is the deviatoric part of the Cauchy stress tensor and $\boldsymbol{\alpha}$ is the back stress tensor. Back stress tensor represents the center of the subsequent yield surfaces geometrically. Also, in equation (4), $\dot{\phi}$ is a proportionality factor which is obtained by the consistency condition. The consistency condition based on the von Mises criterion, $f = 3(\mathbf{S} - \boldsymbol{\alpha}):(\mathbf{S} - \boldsymbol{\alpha})/2 - \sigma_Y^2$, is defined as follows:

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{\sigma}^o + \frac{\partial f}{\partial \boldsymbol{\alpha}} : \boldsymbol{\alpha}^o = 0 \tag{5}$$

where ":" denotes the double contraction, σ_Y is the yield stress and $\boldsymbol{\alpha}^o$ is the corotational rate of the back stress tensor, which relates to the plastic part of the corotational rate of the logarithmic strain tensor through the following kinematic hardening model (evolution equation):

$$\boldsymbol{\alpha}^o = \frac{2}{3} h_\alpha \mathbf{e}^{op} \tag{6}$$

where h_α is the kinematic hardening coefficient. Substitution of equations (6) into equation (5) and using the von Mises yield criterion, yield the proportionality factor, $\dot{\phi}$ as:

$$\dot{\phi} = \frac{9(\mathbf{S} - \boldsymbol{\alpha}) : \boldsymbol{\sigma}^o}{4 h_\alpha \sigma_Y^2} \tag{7}$$

Substitution of equations (3) and (4) into equation (2), and also inserting equation (4) into equation (6), yield the following equations:

$$\mathbf{e}^o = \frac{1}{2G} \boldsymbol{\sigma}^o - \frac{\nu}{E} \mathrm{tr}(\dot{\boldsymbol{\sigma}})\mathbf{I} + \dot{\phi}(\mathbf{S} - \boldsymbol{\alpha}) \tag{8}$$

$$\boldsymbol{\alpha}^o = \frac{2}{3} h_\alpha \dot{\phi}(\mathbf{S} - \boldsymbol{\alpha}) \tag{9}$$

Equations (8) and (9) together with equation (7) make the governing equations for the analysis of an elastic-plastic kinematic hardening material. These equations should numerically be solved for the components of stress and back stress tensors. As an application of the proposed constitutive modelling, for different corotational rates, the simple shear problem is solved and the stress components are plotted versus the shear displacement.

Application in Simple Shear Problem

In two dimensional simple shear problem, the deformation gradient tensor is in the form of:

$$[F]_{ij} = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} \tag{10}$$

where γ is the shear displacement. The logarithmic strain tensor for the simple shear problem is given as follows:

$$[e]_{ij} = \frac{E}{s}\begin{bmatrix} \gamma & 2 \\ 2 & -\gamma \end{bmatrix} \tag{11}$$

where $s = \sqrt{\gamma^2 + 4}$ and $E = \ln(\gamma + s)/2$. Substituting equations (11) and (7) into equations (8) and (9), yields the following governing equations for the simple shear problem, as:

$$\begin{aligned}
&(\frac{1}{2G} + \xi^2\mu)\dot{\sigma} + \xi\zeta\mu\dot{\tau} = \frac{\dot{\gamma}}{s^2}(\gamma + \frac{4E}{s}) - \frac{4\Omega^o E}{s} + \frac{\Omega^o}{G}\tau - \xi\eta\mu \\
&\xi\zeta\mu\dot{\sigma} + (\frac{1}{2G} + \xi^2\mu\zeta^2\mu)\dot{\tau} = \frac{2\dot{\gamma}}{s^2}(1 - \frac{\gamma E}{s}) + \frac{2\Omega^o\gamma E}{s} - \frac{\Omega^o}{G}\sigma - \zeta\eta\mu \\
&\dot{\alpha}_{11} - \frac{2}{3}h_\alpha\xi^2\mu\dot{\sigma} - \frac{2}{3}h_\alpha\xi\zeta\mu\dot{\tau} = \frac{2}{3}h_\alpha\xi\eta\mu + 2\Omega^o\alpha_{12} \\
&\dot{\alpha}_{12} - \frac{2}{3}h_\alpha\xi\zeta\mu\dot{\sigma} - \frac{2}{3}h_\alpha\zeta^2\mu\dot{\tau} = \frac{2}{3}h_\alpha\zeta\eta\mu - 2\Omega^o\alpha_{11}
\end{aligned} \tag{12}$$

where σ, τ, α_{11} and α_{12} are the normal and shear components of stress and back stress tensors, respectively and ξ, ζ, η and μ are defined as follows:

$$\xi = (\sigma - \alpha_{11}), \qquad \zeta = (\tau - \alpha_{12}), \qquad \eta = 2\Omega^o(\zeta\sigma - \xi\tau), \qquad \mu = 9/(4h_\alpha\sigma_Y^2) \tag{13}$$

In equations (12), Ω^{o} is the off-diagonal component of the spin tensor associated with the corresponding corotational rate ,equation (1). In the simple shear problem Ω^{o} for E-, Green-Naghdi and log- corotational rates are as follows, respectively [7,8]:

$$\Omega^{o} = \Omega_{12}^{E} = \frac{\dot{\gamma}}{s^2}, \qquad \Omega^{o} = \Omega_{12}^{GN} = \frac{2\dot{\gamma}}{s^2}, \qquad \Omega^{o} = \Omega_{12}^{\log} = \frac{\dot{\gamma}}{s^2}\left(1 + \frac{\gamma s}{4E}\right) \tag{14}$$

Equations (12) must be solved numerically for the mentioned corotational rates for the unknown components, σ, τ, α_{11} and α_{12}.

Numerical Results and Conclusion

For an elastic-plastic kinematic hardening material with $h_{\alpha} = 0.25G$ and $\sigma_Y = 0.15G$ (G is the modulus of rigidity), equations (12) are solved numerically in terms of the unknown components of stress and back stress tensors, σ, τ, α_{11} and α_{12}, respectively.

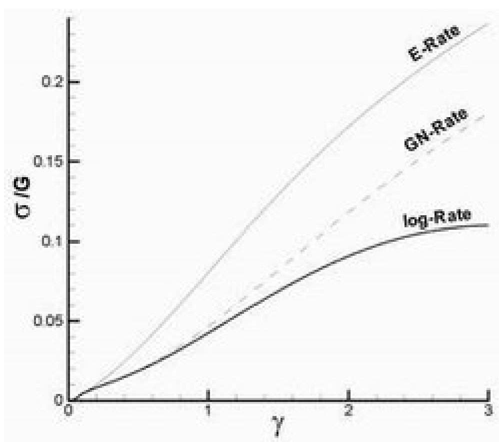

Figure 1- Dimensionless normal stress versus shear displacement for an elastic-plastic, kinematic hardening material with $h_{\alpha} = 0.25G$ and $\sigma_Y = 0.15G$

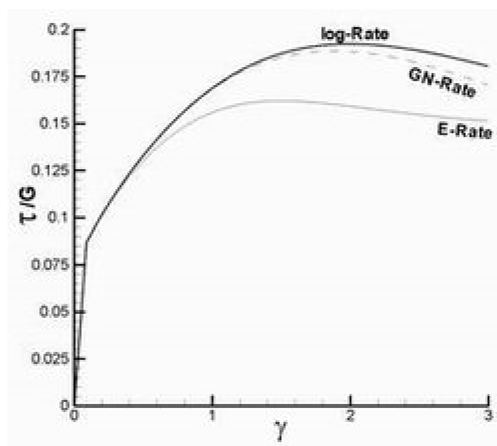

Figure 2- Dimensionless shear stress versus shear displacement for an elastic-plastic, kinematic hardening material with $h_{\alpha} = 0.25G$ and $\sigma_Y = 0.15G$

In figures 1 and 2, dimensionless components of stress tensor, σ / G and τ / G are plotted versus the shear displacement for the mentioned elastic-plastic hardening material for different corotational rates (E-, GN and log-rates). It is noted that the plastic deformation begins at $\gamma \approx 0.18$ and consequently in figures 1 and 2, the elastic parts are small. As it is shown in the figures, different corotational rates make different results. In figure 1, for $\gamma < 1$, and in figure 2, for $\gamma < 2$, the results are the same, except for the E-rate. Results are plotted for $0 < \gamma < 3$.

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