Subharmonic mechanism of the mode C instability

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The perturbation field of the recently discovered subharmonic mode C instability in the wake behind a ring is compared via a side-by-side comparison to the perturbation fields of the modes A and B instabilities familiar from past studies of the vortex street behind a circular cylinder. Snapshots of the wake are presented over a full shedding cycle, along with evidence from a linear stability analysis, to verify and better understand how the subharmonic instability is sustained. © *2005 American Institute of Physics*. [DOI: 10.1063/1.2139682]

Since dye visualizations[1] and Strouhal frequency measurements[2] showed that the transition from laminar periodic flow to turbulent flow in the vortex street behind a circular cylinder occurred through a series of well-defined three-dimensional transitions, much work has followed to determine the nature of these intermediate modes.

Recent work[3–5] has focused on a ring aligned normal to the direction of flow. For rings, an aspect ratio parameter (AR) is defined as the ratio of mean diameter to cross-section diameter. Thus bodies that vary continuously from a sphere at aspect ratio $\mathrm{AR}=0$ towards a straight circular cylinder local to the ring cross section as $\mathrm{AR}\rightarrow\infty$ can be represented. It has been shown[5] that when $\mathrm{AR}\gtrsim 4$, the wake undergoes an axisymmetric Hopf bifurcation at Reynolds numbers below the onset of three-dimensional flow. Thus at these aspect ratios, three-dimensional wake transitions occur through the evolution of instability modes analogous to those observed behind a circular cylinder. As well as predicting instability modes similar in symmetry, spanwise wavelength, and transition Reynolds numbers to the modes A and B instabilities behind a circular cylinder, studies of ring wakes have discovered a third instability known as mode C. This mode is remarkable for two reasons: first, it is predicted to be a true subharmonic mode (the instability exceeds the unit circle through -1 on the real axis), and second, it is predicted to be the first-occurring mode for rings with $4\lesssim\mathrm{AR}\lesssim 8$. The first experimental observation of this mode is revealed in Sheard *et al.*[6]

A review of the key events in the development of our understanding of the three-dimensional transitions in vortex streets best begins with the work by Williamson.[1,2] Williamson observed a wake with a spanwise periodic three-dimensional structure at Reynolds numbers above $\mathrm{Re}\approx 180$, with a spanwise wavelength of 3–4d (where d is the cylinder diameter). This mode, mode A, was observed to be replaced by a second mode, again periodic in the spanwise direction, but adopting a much shorter wavelength of approximately 1d. This second mode, mode B, is observed at Reynolds numbers above $\mathrm{Re}\approx 260$, with remnants persisting to much higher Reynolds numbers, where the flow is fully turbulent.[7,8]

The linear stability analysis of Barkley and Henderson[9] determined that both the modes A and B instabilities occurred through synchronous bifurcations (the Floquet multiplier exceeds the unit circle through +1 on the real axis), and verified the critical Reynolds numbers and preferred spanwise wavelengths of the instabilities. The later three-dimensional computations of the cylinder wake[10,11] verified that the Floquet stability analysis technique provided accurate predictions of the spanwise wavelength, symmetry, and mode topology of the corresponding three-dimensional wakes. Modes A and B were predicted to occur above $\mathrm{Re}=188.5$ and $\mathrm{Re}=259$, with spanwise wavelengths 3.96d and 0.8d, respectively. Three-dimensional computations by Thompson, Hourigan, and Sheridan[12] elucidated the streamwise vortical structure of the instabilities, and a further detailed stability analysis[13] suggested that the mode A instability may be initiated by an elliptic instability in the vortex rollers.

For the wake of a circular cylinder, modes with spanwise wavelengths between those of the modes A and B instabilities have been shown to remain stable and contain an imaginary component.[9,14] An imaginary component implies a quasiperiodic instability, leading to standing- or traveling-wave modes.

For the unperturbed[15] and perturbed[16–18] wakes behind circular cylinders, and the wakes behind square cylinders[14,19] and rings,[3] the literature reveals several examples of unstable three-dimensional modes in the intermediate spanwise wavelength range of approximately 1.5d–2d. For a circular cylinder, early studies[15,19] suggested that these three-dimensional modes exhibited subharmonic properties (i.e., the three-dimensional features alternated in sign from one shedding cycle to the next) causing a doubling of the effective period of the wake. However, the detailed stability analyses conducted for circular[9] and square[14] cylinders proved that the earlier classification of these modes as subharmonic[19] was erroneous. In fact, these modes are quasiperiodic, with a small imaginary component and large negative real component. This gave the appearance of a subharmonic mode, explaining their incorrect classification. For vortex streets with Z_2 symmetry (a $T/2$ time shift and spatial reflection about the wake center line recovers the original flow) it has been speculated[14] that the generic bifurcation scenario to three-dimensional flow comprises the familiar real mode A and mode B instabilities, with wavelengths approximately 4 and 1 times an appropriate body length scale (e.g., diameter d),

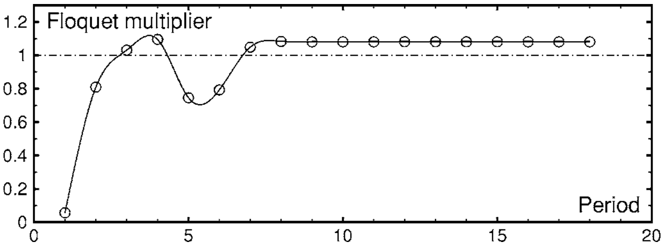

FIG. 1. Floquet multiplier convergence for the mode C instability with $m=10$ at Re=170 with AR=5.

as well as an additional quasiperiodic instability with a spanwise wavelength approximately twice the body length scale.

The wakes behind open rings do not possess a Z_2 symmetry due to the ring curvature, but as $\mathrm{AR}\rightarrow\infty$ the ring locally approaches a straight circular cylinder, and the wake approaches a Z_2 symmetry state. Therefore it has been suggested[3] that the real subharmonic mode C instability emerges as a result of the asymmetry about the wake center line. It was recently implied[20] that the predicted mode C instability behind rings might have been a quasiperiodic mode due to the spanwise phase of the perturbation field being locked in the original stability analysis.[3] In fact, the method employed in this study *can* distinguish subharmonic (real) modes from quasiperiodic (complex) modes, based on the Floquet multiplier behavior, and the characterization of the mode C instability as a subharmonic mode is reinforced in this paper.

In this paper, the existence and classification of the subharmonic mode C instability is proved by rigorous examination of the results of a linear stability analysis, and is verified through careful and accurate computations. The wake of a ring with AR=5 at Re=170 is studied, as it is near to (and slightly above) the critical Reynolds number for the emergence of the mode C instability with azimuthal mode number $m=10$ and a spanwise wavelength of approximately $1.6d$. This Reynolds number is below the onset of the modes A and B instabilities; hence predictions pertaining to mode C are relevant to the physical wake. In addition, the three instability modes are compared over a full shedding cycle to identify features distinguishing the mode C instability from the familiar modes A and B instabilities.

The axisymmetric flow was computed using a spectral-element method for the incompressible Navier-Stokes equations. The same formulation of the code has previously been applied to accurately compute the wake behind a sphere,[21] and the wake behind rings.[3] The perturbation fields for the three-dimensional instability modes were computed using a linear Floquet stability analysis technique.[3,9] In this technique, a small three-dimensional velocity perturbation of fixed azimuthal wavelength is computed using linearized three-dimensional Navier-Stokes equations in addition to the periodic two-dimensional base flow. The evolution of this perturbation field is monitored to determine the stability of the base flow.

Figure 1 shows the convergence of the Floquet multiplier for the Floquet mode with $m=10$ at Re=170, the fastest-growing azimuthal mode of the mode C instability. The computation was stopped when the dominant mode was isolated in the perturbation field (i.e., constant or oscillating Floquet multipliers for real and complex modes, respectively). In this case, the observed multiplier convergence provided confirmation that the mode contained no imaginary component, as this behavior is exclusive to real modes.[14] In addition, the alternation in sign of the perturbation field each period [e.g., compare Figs. 2(c)(i) and 2(c)(v)] verifies that the mode is subharmonic. As an independent check of this result, the sum of these perturbation fields was evaluated, and a zero field resulted, to the limit of numerical accuracy.

Figure 2 shows, for the first time, a comparison between the perturbation fields generated over a full shedding cycle for each of the modes A, B, and C instabilities. Flow fields are presented at quarter-period intervals [parts (i) to (v) representing times from $0.0T$ to $1.0T$], and the shedding cycles are initiated from the point of maximum pressure drag. Computed at Re=200 and Re=320, respectively, the modes A and B presented here are similar in structure and symmetry to the equivalent modes predicted in the wake behind a circular cylinder.[9,13] Note that unlike modes A and B, the sign of the mode C perturbation vorticity field alternates over the shedding cycle between Figs. 2(c)(i) and 2(c)(v). The structure of the mode C perturbation field is a curious hybrid, with features similar to both modes A and B. The near wake of the mode C instability (i.e., within approximately $3d$ of the ring) bears a strong resemblance to the mode B instability, with strong vorticity present in the braid region between base flow vortices. Further downstream, the perturbation field resembles that of the mode A instability, with perturbation vorticity localized within the base flow vortices.

The base flow vorticity contours in Fig. 2 show that the asymmetry induced on the vortex street by the curvature of the ring has the effect of pairing each lower vortex roller to a previously shed upper roller. This phenomenon was observed in studies of the vortex shedding behind rings,[22,23] and relates to the asymmetry induced by the curvature of the ring. This vortex pairing is driven by the extended attachment duration of the upper vortex (with negative vorticity) to the ring, and a streamwise elongation of the vortex. A rapid evolution of a strong (positive vorticity) lower vortex roller occurs over a time $0.75T\lesssim t\lesssim 1.0T$, after which the vortex pair detach and convect downstream.

Initially, both the upper and lower vortices are strained [observe the pair of vortex rollers located approximately $3d$ downstream in Fig. 2(c)(iv) at $t=0.75T$). As this pair of vortices convect downstream (e.g., $t=1.0T$, Fig. 2(c)(v)], the upper vortex becomes circular while the lower vortex remains strained in an elliptical shape. The perturbation field vorticity in these shed vortex pairs is consistent with an elliptic instability, which manifests itself as a counterrotating perturbation vortex pair in a strained elliptical base flow vortex (see Refs. 13 and 24–26 for detailed discussion of elliptic instabilities in strained vortices). Due to higher strain, the elliptic instability is stronger in the lower vortex [see Figs. 2(c)(i) and 2(c)(v)].

Consider closely the perturbation fields of the mode C instability in Figs. 2(c)(i)–2(c)(v). At time $t=0.0T$, a counterrotating pair of perturbation field vortices can be observed in

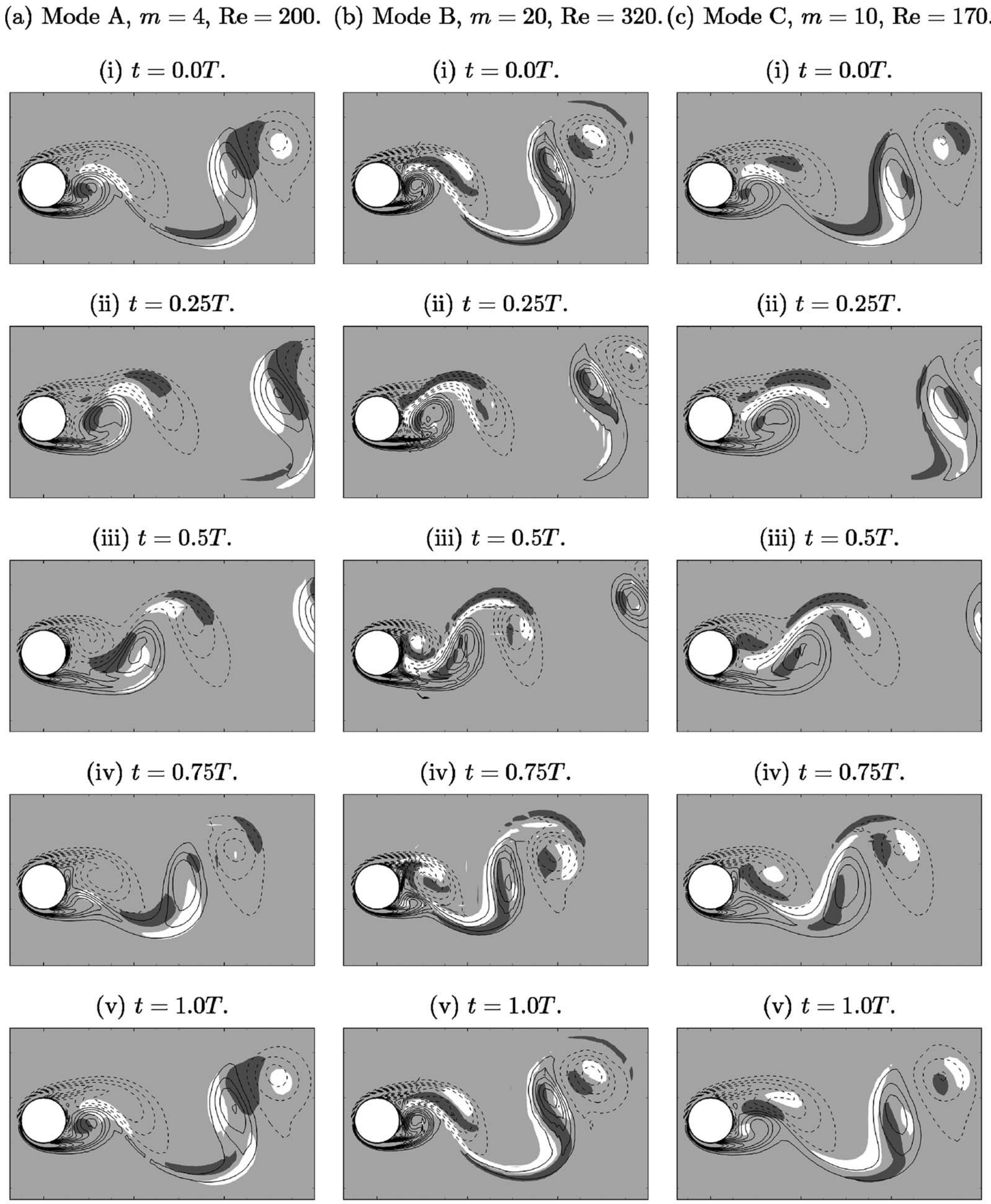

FIG. 2. A comparison between the perturbation fields obtained during one shedding cycle for each of the modes A, B, and C instabilities [(a), (b), and (c), respectively]. Arbitrary contour lines show the base flow vorticity (dashed lines denote negative vorticity), and positive and negative spanwise vorticity in the perturbation field is shaded white and black, respectively.

the detaching upper roller approximately $1d$ behind the ring cross section. At first glance, this appears similar to the perturbation field vortices in the mode A wake at $t=0.25T$. One difference may be observed, though. For the mode A wake in Fig. 2(a)(ii), this pair of perturbation field vortices are contained within the detaching upper roller, whereas in the mode C wake, one of the counter-rotating vortices is located between the detaching upper roller and the forming attached lower roller. By following the progression of the perturbation field vorticity of the mode C instability over the full shedding cycle, it can be observed that the sign of the vorticity present between the rollers is opposite to the sign of the vorticity within the upper roller [Figs. 2(c)(i) and 2(c)(ii)] and the lower roller [Figs. 2(c)(ii) and 2(c)(iii)]. As the lower roller is cast off the rear of the ring in Fig. 2(c)(iv), the perturbation vorticity that was present between the rollers convects downstream, and the opposite-sign vorticity in the newly forming upper roller [Figs. 2(c)(iii)–2(c)(iv)] shifts from within the roller to the now vacant region between the attached rollers. This process repeats and thus the alternation in sign of the perturbation field each period is sustained.

The earlier classification of the mode C instability in ring wakes as a subharmonic mode has been verified here. In addition, the perturbation fields of the three dominant instability modes behind rings have been presented for side-by-side comparison.

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[1]C. H. K. Williamson, "The existence of two stages in the transition to three-dimensionality of a cylinder wake," Phys. Fluids **31**, 3165 (1988).
[2]C. H. K. Williamson, "Defining a universal and continuous Strouhal-Reynolds number relationship for the laminar vortex shedding of a circular cylinder," Phys. Fluids **31**, 2742 (1988).
[3]G. J. Sheard, M. C. Thompson, and K. Hourigan, "From spheres to circular cylinders: The stability and flow structures of bluff ring wakes," J. Fluid Mech. **492**, 147 (2003).
[4]G. J. Sheard, M. C. Thompson, and K. Hourigan, "Asymmetric structure and non-linear transition behaviour of the wakes of toroidal bodies," Eur. J. Mech. B/Fluids **23**, 167 (2004).
[5]G. J. Sheard, M. C. Thompson, and K. Hourigan, "From spheres to circular cylinders: Non-axisymmetric transitions in the flow past rings," J. Fluid Mech. **506**, 45 (2004).
[6]G. J. Sheard, M. C. Thompson, K. Hourigan, and T. Leweke, "The evolution of a subharmonic mode in a vortex street," J. Fluid Mech. **534**, 23 (2005).
[7]C. H. K. Williamson, J. Wu, and J. Sheridan, "Scaling of streamwise vortices in wakes," Phys. Fluids **7**, 2307 (1995).
[8]J. Wu, J. Sheridan, M. C. Welsh, and K. Hourigan, "Three-dimensional vortex structures in a cylinder wake," J. Fluid Mech. **312**, 201 (1996).
[9]D. Barkley and R. D. Henderson, "Three-dimensional Floquet stability analysis of the wake of a circular cylinder," J. Fluid Mech. **322**, 215 (1996).
[10]R. D. Henderson and D. Barkley, "Secondary instability in the wake of a circular cylinder," Phys. Fluids **8**, 1683 (1996).
[11]R. D. Henderson, "Non-linear dynamics and pattern formation in turbulent wake transition," J. Fluid Mech. **352**, 65 (1997).
[12]M. C. Thompson, K. Hourigan, and J. Sheridan, "Three-dimensional instabilities in the wake of a circular cylinder," Exp. Therm. Fluid Sci. **12**, 190 (1996).
[13]M. C. Thompson, T. Leweke, and C. H. K. Williamson, "The physical mechanism of transition in bluff body wakes," J. Fluids Struct. **15**, 607 (2001).
[14]H. M. Blackburn and J. M. Lopez, "On three-dimensional quasiperiodic Floquet instabilities of two-dimensional bluff body wakes," Phys. Fluids **15**, L57 (2003).
[15]G. E. Karniadakis and G. S. Triantafyllou, "Three-dimensional dynamics and transition to turbulence in the wake of bluff objects," J. Fluid Mech. **238**, 1 (1992).
[16]B. R. Noack, R. König, and H. Eckelmann, "Three-dimensional stability analysis of the periodic flow around a circular cylinder," Phys. Fluids A **5**, 1279 (1993).
[17]B. R. Noack and H. Eckelmann, "A global stability analysis of the steady and periodic cylinder wake," J. Fluid Mech. **270**, 297 (1994).
[18]B. R. Noack and H. Eckelmann, "A low-dimensional Galerkin method for the three-dimensional flow around a circular cylinder," Phys. Fluids **6**, 124 (1994).
[19]J. Robichaux, S. Balachandar, and S. P. Vanka, "Three-dimensional Floquet instability of the wake of a square cylinder," Phys. Fluids **11**, 560 (1999).
[20]G. E. Karniadakis and S. J. Sherwin, *Spectral/hp Element Methods for Computational Fluid Dynamics* (Oxford University Press, New York, 2005).
[21]M. C. Thompson, T. Leweke, and M. Provansal, "Kinematics and dynamics of sphere wake transition," J. Fluids Struct. **15**, 575 (2001).
[22]D. R. Monson, "The effect of transverse curvature on the drag and vortex shedding of elongated bluff bodies at low Reynolds number," ASME Trans. J. Fluids Eng. **105**, 308 (1983).
[23]T. Leweke and M. Provansal, "The flow behind rings: Bluff body wakes without end effects," J. Fluid Mech. **288**, 265 (1995).
[24]B. J. Bayly, "Three-dimensional instability of elliptical flow," Phys. Rev. Lett. **57**, 2160 (1986).
[25]M. J. Landman and P. G. Saffman, "The three-dimensional instability of strained vortices in a viscous fluid," Phys. Fluids **30**, 2339 (1987).
[26]C. H. K. Williamson, "Mode A secondary instability in wake transition," Phys. Fluids **8**, 1680 (1996).